Principles of Math 12 Workbook

- Transformations
- Exponential and Logarithmic Functions
- Geometric Series and Applications of Exponential and Logarithmic Functions
- Trigonometry - Functions and Graphs
- Trigonometry - Equations, Identities, and Modelling
- Probability
- Permutations and Combinations
- Statistics and Probability
- Conic Sections (Quadratic Relations)

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Greg Ranieri

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**About the Principles of Math 12 Workbook**

The Principles of Math 12 Workbook is a complete resource for the Principles of Mathematics 12 Curriculum. Each curricular topic is subdivided into individual lessons. Most lessons can be covered in one hour (plus homework time), but some may require more time to complete. Lessons are composed of four parts:

- **Warm-Ups** - which could be review, preview or investigative work.
- **Class Examples** - which are intended to be teacher led.
- **Assignments** - short response, extended response, multiple choice and numeric response questions are provided for student practice.
- **Answer Key** - answers to the assignment questions.

The **Teacher Solution Manual** is a copy of the workbook with detailed solutions to all the Warm-ups, Class Examples, and Assignments.

The **Student Solution Manual** contains detailed solutions to all the Warm-ups, Class Examples, and Assignments without the questions.

The material has been piloted in several schools and colleges and adjustments have been made based on student and teacher feedback.

Every effort has been made to achieve a high standard of accuracy in the writing of the workbook, solution manual and answer key. We accept full responsibility for any errors and welcome feedback.

**Acknowledgments**

We would like to acknowledge the following people for their contributions in the production of this workbook:

- William Stee, Leona Fry, June Wong, Baubach Naderi, Linda Binder, Cori Cowan, and other colleagues for their feedback and suggestions.
- our students for their suggestions, opinions, and encouragement.
- Tony Audia for his support.

Most of all, we would like to thank our families, especially our respective wives, Susan, Linda, and Rose, for their patience and understanding.
### Advantages for Students

- Students write in the workbook so that the math theory, worked examples, and assignments are all in one place for easy review.
- Students can write on the diagrams and graphs.
- Provides class examples and assignments so that students can use their time more efficiently by focusing on solving problems and making their own notes.
- For independent learners the workbook plus solution manual fosters self-paced learning.
- Encourages group learning and peer tutoring.
- The design of the workbook ensures that students are fully aware of the course expectations.
- We hope you enjoy using this workbook and that with the help of your teacher you realize the success that thousands of students each year are achieving using the workbook series.

### Advantages for Teachers

- Written by teachers experienced in preparing students for success in diploma examinations.
- Comprehensively covers the Principles of Mathematics 12 curriculum including recent updates.
- Can be used as the main resource, or in conjunction with a textbook, or for extra assignments or review.
- Lessons have been thoroughly piloted in the classroom and modified based on student and teacher feedback.
- The workbook is continually updated to take account of major and minor curriculum changes.
- Reduces school photocopying costs and time.
- Allows for easy lesson planning in the case of teacher or student absence.

Teacher, student, and parent responses have been very positive. We welcome your feedback. It enables us to produce a high quality resource meeting our goal of success for both teachers and students.
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Transformations Lesson #1: Functions and Graphs Review

Warm-Up #1

Introduction

In this unit we will use graphs of functions learned in previous math courses, namely:

- Polynomial Functions
- Radical Functions
- Absolute Value Functions
- Rational Functions

This lesson is a review of some of the properties of these functions.

Polynomial Functions

A polynomial function in $x$ is a function in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where

- $a_0, a_1, a_2, \ldots, a_n$ are real numbers, $a_n \neq 0$,
- $n \in \mathbb{N}$.

$a_1, a_2, \ldots, a_n$ are called coefficients. $a_n$ is called the leading coefficient and $a_0$ is the constant term. The value of $n$ is the degree of the polynomial.

For example, the polynomial function $f(x) = 7x^3 + x^4 - 8x^2 + 5$ has

degree_______, leading coefficient _______, and constant term _______

Three common polynomial functions we will use for transformations are

- Linear Functions
- Quadratic Functions
- Cubic Functions

Linear Functions

A linear function is a polynomial function of degree 1 of the form $f(x) = ax + b$.

---

Sketch each of the following linear functions and answer the questions which follow.

i) $y = 3x + 1$

a) What is the domain?

b) What is the range?

c) What is another name for the function in ii)?

ii) $y = 5$
Quadratic Functions

A quadratic function is a polynomial function of degree 2 which can be written in general or standard form.

A quadratic function written in the form
\[ f(x) = ax^2 + bx + c \]
where \( a \neq 0 \) is called the general form.

A quadratic function or equation written in the form
\[ f(x) = a(x - p)^2 + q \]
where \( a \neq 0 \) is called the standard form.

Class Ex. #2

a) Use a graphing calculator to sketch the graph of
\( y = x^2 - 3x - 18 \) and determine:
   i) the zeros
   ii) the y-intercept
   iii) the coordinates of the vertex
   iv) the domain and range.

b) Use factoring to determine the zeros.

c) Rewrite the quadratic function in standard form.
   Explain how this form helps determine the coordinates of the vertex.

Class Ex. #3

Use a graphing calculator to sketch the graph of the quadratic function
\( f(x) = -3x^2 + 4x + 1 \).

a) Use the features of a graphing calculator to determine the zeros to the nearest hundredth.

b) Use the quadratic formula to determine the exact values of the zeros in simplest radical form.
Cubic Functions

A cubic function is a polynomial function of degree 3 of the form
\[ f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0. \]

Use a graphing calculator to sketch the graph of the cubic function
\[ y = x^3 - 8x^2 + 16x - 8. \]

a) Determine the zeros to the nearest tenth.

b) Determine the exact value of the zeros.

Absolute Value Function

An absolute value function is a function of the form \( f(x) = |x| \), where
\[ f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

Sketch the graph of the absolute value function \( y = |x - 2| \) and determine the domain and range.

Radical Functions

A radical function is a function which contains a variable in the radicand such as \( f(x) = \sqrt{x} \).

Sketch the radical function \( y = \sqrt{2x - 1} \).
Determine the domain and range of the function.
**Rational Functions**

Rational functions are functions of the form \( f(x) = \frac{n(x)}{d(x)} \) where \( n(x) \) and \( d(x) \) are polynomials and \( d(x) \neq 0 \).

**Asymptote** - Some rational functions have an **asymptote** - an imaginary line to which the extreme ends of a graph approaches closer and closer as \(|x| \) or \(|y| \) increases.

Asymptotes are represented by dotted lines on the graph of a function. They are not part of the graph of a function, but help to show the nature of the function.

The dotted lines for an asymptote will not appear when the calculator is in dot mode or decimal connected mode. In the Greek language asymptote means “not meeting”

**Point of Discontinuity** - Others rational functions have a **point of discontinuity** - a break in the graph. The point of discontinuity can be seen using the Zoom Decimal window (or multiple of the Zoom Decimal window).

---

Use a graphing calculator to sketch the graph of each of the following rational functions. State the domain, the range, the equations of any asymptotes, and the coordinates of any points of discontinuity.

a) \( y = \frac{1}{x + 2} \) 

b) \( y = \frac{x + 2}{x - 3} \) 

c) \( y = \frac{x^2 + 3x + 2}{x + 2} \)

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**Complete Assignment Questions #1 - #11**
Assignment

1. Graph each of the following functions and determine:
   
i) the zeros (to the nearest tenth if necessary)
   
 ii) the domain and range

   a) \( y = -2x + 4 \)  
   
   b) \( y = 5x - 10 \)  
   
   c) \( y = x^2 - 2x - 15 \)
   
   d) \( y = -x^2 - 3x - 4 \)  
   
   e) \( y = (x + 1)(x + 2)(x - 5) \)
   
2. The linear function \( y = ax + b \) is in slope y-intercept form.
   
a) Which parameter represents the slope of the line?
   
b) Which parameter represents the y-intercept?
   
c) If \( a > 0 \), describe the slope.
   
d) If \( a < 0 \), describe the slope.

3. How can you tell from the quadratic function \( y = ax^2 + bx + c \) whether the graph of the function will open up or open down?
4. Use a graphing calculator to sketch the graph of the quadratic function
   \[ f(x) = 2x^2 - 5x - 6. \]
   a) Use the features of a graphing calculator to determine the zeros to the nearest hundredth.
   b) Use the quadratic formula to determine the exact values of the zeros in simplest radical form.

5. Sketch the graph of the following functions and state the domain and range.
   a) \( y = |x| + 4 \)
   b) \( y = |x + 3| \)
   c) \( y = -|x + 2| \)
   d) \( y = -|x| - 3 \)

6. Which of the graphs in Question #5 open up? Open down?
   What determines the direction of opening?

7. Sketch the graph of the following functions and state the domain and range.
   What shape is each graph?
   a) \( y = \sqrt{16 - x^2} \)
   b) \( y = -\sqrt{16 - x^2} \)
8. Combining the equations from Question #7a and #7b results in \( y = \pm \sqrt{16 - x^2} \).
   a) Square both sides of the equation to rewrite in the form \( x^2 + y^2 = k \).

   b) What geometrical shape is formed by combining the graphs from question #7a and #7b?

   c) How does the value of \( k \) relate to the radius of the circle?

   d) Without using a graphing calculator, sketch the graph of \( x^2 + y^2 = 49 \).

9. Use a graphing calculator to sketch the graph of each of the following rational functions. State the domain, the range, the equations of any asymptotes, and the coordinate of any points of discontinuity.
   a) \( y = \frac{2}{x - 4} \)  
   b) \( y = \frac{x}{x + 1} \)  
   c) \( y = \frac{x^2 - 9x + 20}{x - 5} \)

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10. The function \( f \) is defined by \( f(x) = 2x^2 + 3 \). Which of the following represents \( f(2x) \)?
   A. \( 2x^2 + 3 \)  
   B. \( 4x^2 + 3 \)  
   C. \( 8x^2 + 3 \)  
   D. \( 4x^2 + 6 \)  

11. The rational expression \( r(x) = \frac{x^2 - 8x + 12}{x - 6} \) has a point of discontinuity. The value of the \( y \)-coordinate of the point of discontinuity, to the nearest whole number, is _____.

---

**Answer Key**

1. a) zeros: 2  
   Domain: \( \mathbb{R} \)  
   Range: \( \mathbb{R} \)  
   b) zeros: 2  
   Domain: \( \mathbb{R} \)  
   Range: \( \mathbb{R} \)  
   c) zeros: \(-3, 5\)  
   Domain: \( \mathbb{R} \)  
   Range: \( \{y \mid y \geq -16, y \in \mathbb{R}\} \)  
   d) zeros: none  
   Domain: \( \mathbb{R} \)  
   Range: \( \{y \mid y \leq -\frac{7}{4}, y \in \mathbb{R}\} \)  
   e) zeros: \(-1, -2, 5\)  
   Domain: \( \mathbb{R} \)  
   Range: \( \mathbb{R} \)

2. a) a  
   b) b  
   c) rises from left to right  
   d) falls from left to right  

3. If \( a > 0 \), the graph opens up. If \( a < 0 \), the graph opens down.

4. a) \(-0.89, 3.39\)  
   b) \( x = \frac{5 \pm \sqrt{73}}{4} \)

5. a) Domain: \( \mathbb{R} \)  
   Range: \( \{y \mid y \geq 4, y \in \mathbb{R}\} \)  
   b) Domain: \( \mathbb{R} \)  
   Range: \( \{y \mid y \geq 0, y \in \mathbb{R}\} \)  
   c) Domain: \( \mathbb{R} \)  
   Range: \( \{y \mid y \leq -3, y \in \mathbb{R}\} \)  
   d) Domain: \( \mathbb{R} \)  
   Range: \( \{y \mid y \leq -3, y \in \mathbb{R}\} \)

6. • Graphs a and b open up and graphs c and d open down.  
   • If there is a negative sign outside the absolute value symbol, the graph opens down.  
   If there is no negative sign outside the absolute value symbol, the graph opens up.

7. a) Domain: \( \{x \mid -4 \leq x \leq 4, x \in \mathbb{R}\} \)  
   Range: \( \{y \mid 0 \leq y \leq 4, y \in \mathbb{R}\} \)  
   b) Domain: \( \{x \mid -4 \leq x \leq 4, x \in \mathbb{R}\} \)  
   Range: \( \{y \mid -4 \leq y \leq 0, y \in \mathbb{R}\} \)

8. a) \( x^2 + y^2 = 16 \)  
   b) circle  
   c) \( k = r^2 \)  
   d) circle centre \((0, 0)\) with radius 7

9.

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10. C  
11. 4
Transformations Lesson #2:  
Function Notation and Inverse Functions

**Function Notation**

If \( f(x) = x^2 + 4x + 5 \), find the following in simplest form:

- **a)** \( f(2) \)
- **b)** \( 2f(x) \)
- **c)** \( f(2x) \)
- **d)** \( f(x) + 2 \)
- **e)** \( f(x - 2) \)
- **f)** \( f(-x) \)

If \( f(x) = |x| \), write the following in terms of the function \( f \).

- **a)** \( |x| - 4 \)
- **b)** \( |x - 4| \)
- **c)** \( 2|x| \)
- **d)** \( |2x| \)
- **e)** \( 3 - |x| \)
- **f)** \( |x + 3| - 2 \)
- **g)** \( \frac{1}{|x|} \)
- **h)** \( |-x| \)

If \( f(x) = x^3 \), write the following in terms of the function \( f \).

- **a)** \( x^3 + 1 \)
- **b)** \( (x + 1)^3 \)
- **c)** \( 4x^3 \)
- **d)** \( (4x)^3 \)
- **e)** \( 2x^3 + 3 \)
- **f)** \( 2(x^3 + 3) \)
- **g)** \( -x^3 + 2 \)
- **h)** \( -(x + 4)^3 \)
Inverse of a Function

A function relates elements of the domain to elements of the range. An inverse function reverses this process, i.e. the domain of an inverse function is the range of the original function and the range of the inverse function is the domain of the original function.

- If a function is defined by a set of ordered pairs, the inverse function is defined by interchanging the $x$ and $y$ the coordinates.
- If a function is defined by an equation, the inverse can be determined using the following procedure:
  1. interchange $x$ and $y$
  2. solve for $y$
- If a function is defined graphically, the inverse can be determined by a reflection in the line $y = x$
- The inverse of the function $y = f(x)$ has the notation $y = f^{-1}(x)$, which is equivalent to $x = f(y)$.

Class Ex. #4

a) Find the inverse of the following set of ordered pairs.
\[
\{(−4,−2), (−2,−1), (−1, 0), (0,1), (2,4), (3,8)\}
\]

b) Graph the original set of ordered pairs and the inverse. Join each set of ordered pairs with a smooth curve.

c) Sketch the line of reflection. What is the equation of this line?

Class Ex. #5

a) Find $f^{-1}(x)$ if $f(x) = 3x + 2$.

b) If $f(x) = x^2 - 1$, find $x = f(y)$ and solve for $y$.

Complete Assignment Questions #1 - #6
Assignment

1. If $f(x) = x^3 - 5$, find the following in simplest form:
   a) $f(-1)$
   b) $4f(x)$
   c) $f(4x)$
   d) $f(x) + 5$
   e) $f(x + 5)$
   f) $f(-x)$

2. If $f(x) = \sqrt{x}$, write the following in terms of the function $f$.
   a) $\sqrt{x} - 1$
   b) $\sqrt{x + 3}$
   c) $\sqrt{2x - 1}$
   d) $-3\sqrt{x}$

3. If $f(x) = x^2$, write the following in terms of the function $f$.
   a) $x^2 + 3$
   b) $(x + 3)^2$
   c) $3x^2$
   d) $(3x)^2$
   e) $4x^2 - 7$
   f) $4(x^2 - 7)$
   g) $-2x^2 - 1$
   h) $(-x + 4)^2$
   i) $-3(-x - 2)^2$

4. a) Graph the function $f(x) = x^2 + 4$.
   b) Graph the inverse of $f(x)$.
   c) Find the equation of the inverse function in the form $x = f(y)$ and solve for $y$. 
5. A function \( f \) is defined by \( y = 2x + 1 \). Which of the following is the equation of the inverse of \( f \)?

A. \( x = \frac{y - 1}{2} \)
B. \( x = 2y + 1 \)
C. \( y = \frac{x + 1}{2} \)
D. \( y = 2x - 1 \)

6. The graph of the function \( y = f(x) \) is shown in the diagram below.

Which of the following represents \( f^{-1}(x) \)?

i. 
ii. 
iii. 
iv. 

A. i 
B. ii 
C. iii 
D. iv

Answer Key
1. a) \(-6\)  b) \(4x^3 - 20\)  c) \(64x^3 - 5\)  d) \(x^3\)  e) \(x^3 + 15x^2 + 75x + 120\)  f) \(-x^3 - 5\)
2. a) \(f(x) - 1\)  b) \(f(x + 3)\)  c) \(f(2x - 1)\)  d) \(-3f(x)\)
3. a) \(f(x) + 3\)  b) \(f(x + 3)\)  c) \(3f(x)\)  d) \(f(3x)\)  e) \(4f(x) - 7\)  f) \(4(f(x) - 7)\)  g) \(-2f(x) - 1\)  h) \(f(-x + 4)\)  i) \(-3f(-x - 2)\)
4. c) \(x = y^2 + 4, \ y = \pm\sqrt{x - 4}\)  5. B  6. C
Transformations Lesson #3: Horizontal and Vertical Translations Part 1

Transformations

A transformation is an operation which moves (or maps) a figure from an original position to a new position. Transformations we will consider are translations, reflections, expansions and compressions, reciprocal transformations, and absolute value transformations.

Warm-Up #1 Comparing the Graphs of \( y = f(x) \) and \( y - k = f(x) \)  [or \( y = f(x) + k \)]

Part 1

a) Complete the table of values. The first one has been completed.

| \( x \) | \( y = x^2 \) | \( x \) | \( y = x^2 \) | \( x \) | \( y = x^2 \) |
|---|---|---|---|---|
| -3 | 9 | -3 | 9 | -3 |
| -2 | 4 | -2 | 4 | -2 |
| -1 | 1 | -1 | 1 | -1 |
| 0  | 0 | 0  | 0  | 0  |
| 1  | 1 | 1  | 1  | 1  |
| 2  | 4 | 2  | 4  | 2  |
| 3  | 9 | 3  | 9  | 3  |

b) Use the table of values in a) to graph and label each of the functions on the grid.

c) What is the effect of the parameter \( k \) on the graph of the parabola \( y - k = x^2 \)?

Part 2

a) Use a graphing calculator to graph the following functions:

i) \( y = |x| \)  
ii) \( y = |x| + 2 \)  
iii) \( y = |x| - 3 \)

b) What is the effect of the parameter \( k \) on the graph of \( y = |x| + k \)?

c) What is the effect of the parameter \( k \) on the graph of the function \( y = f(x) + k \)?

d) What is the effect of the parameter \( k \) on the graph of the function \( y - k = f(x) \)?

e) Compared to the graph of \( y = f(x) \), the graph of \( y - k = f(x) \) results in a __________ shift of \( k \) units. If \( k > 0 \), the graph moves _______. If \( k < 0 \), the graph moves _______.

The notation \( y - k = f(x) \) is often used instead of \( y = f(x) + k \) to emphasize that this is a transformation on \( y \). The concept of replacing \( y \) by \( y - k \) will be very important in this course.
Part 1

a) Complete the table of values. The first one has been completed.

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
x & y = x^2 & \hline
-4 & 16 & \hline
-3 & 9 & \hline
-2 & 4 & \hline
-1 & 1 & \hline
0 & 0 & \hline
1 & 1 & \hline
2 & 4 & \hline
3 & 9 & \hline
4 & 16 & \hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\hline
x & y = (x-3)^2 & \hline
-1 & 4 & \hline
0 & 1 & \hline
1 & 2 & \hline
2 & 3 & \hline
3 & 4 & \hline
4 & 5 & \hline
5 & 6 & \hline
6 & 7 & \hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\hline
x & y = (x+3)^2 & \hline
-4 & -1 & \hline
-3 & -6 & \hline
-2 & -5 & \hline
-1 & -4 & \hline
0 & -3 & \hline
1 & -4 & \hline
2 & -5 & \hline
3 & -6 & \hline
\end{array}
\]

b) Use the table of values in a) to graph and label each of the functions on the same grid.

c) What is the effect of the parameter \( h \) on the graph of the parabola \( y = (x-h)^2 \)?

Part 2

a) Use a graphing calculator to graph the following functions:

\[
\begin{align*}
\text{i) } \quad y &= \sqrt{x} \\
\text{ii) } \quad y &= \sqrt{x-4} \\
\text{iii) } \quad y &= \sqrt{x+2}
\end{align*}
\]

b) What is the effect of the parameter \( h \) on the graph of \( y = \sqrt{x-h} \)?

c) What is the effect of the parameter \( h \) on the graph of the function \( y = f(x-h) \)?

d) Compared to the graph of \( y = f(x) \), the graph of \( y = f(x-h) \) results in a __________ shift of \( h \) units. If \( h > 0 \), the graph moves ________. If \( h < 0 \) the graph moves ________.
Translations

A translation is a transformation which slides each point of a figure the same distance in the same direction.

Given the function \( y = f(x) \):

- replacing \( y \) with \( y - k \), \( (i.e. \ y \rightarrow y - k) \) describes a vertical translation.
  
  \[ y - k = f(x) \quad \text{or} \quad y = f(x) + k \]
  describes a vertical translation.

- replacing \( x \) with \( x - h \), \( (i.e. \ x \rightarrow x - h) \) describes a horizontal translation.
  
  \[ y = f(x - h) \]
  describes a horizontal translation.

In general, if

\[ y - k = f(x - h) \]

or

\[ y = f(x - h) + k \]

then

- \( k > 0 \) the graph moves up \( \uparrow \)
- \( k < 0 \) the graph moves down \( \downarrow \)
- \( h > 0 \) the graph moves right \( \rightarrow \)
- \( h < 0 \) the graph moves left \( \leftarrow \)

- If the graph of \( y = f(x) \) is transformed to the graph of \( y - 2 = f(x - 3) \), the replacements for \( x \) and \( y \) are \( x \rightarrow x - 3 \) and \( y \rightarrow y - 2 \). Under this transformation, all points on the graph of \( y = f(x) \) will move 3 units to the right and 2 units up. The point with coordinates \((4, 6)\) will be translated to the point \((7, 8)\). In general the point with coordinates \((x, y)\) is translated to the point \((x + 3, y + 2)\). This translation can be represented in mapping notation by \((x, y) \rightarrow (x + 3, y + 2)\).
- Do not confuse mapping notation with the notation we have used for replacements.

---

**Class Ex. #1**

Describe how the graphs of the following functions relate to the graph of \( y = f(x) \).

a) \( y = f(x - 3) \)  
b) \( y = f(x) + 4 \)  
c) \( y - 1 = f(x + 10) \)

---

**Class Ex. #2**

The point \((2, -3)\) lies on the graph of \( y = f(x) \). State the coordinates of the image of this point under the following transformations.

a) \( y + 8 = f(x) \)

b) \( y = f(x - 7) + 5 \)

c) \((x, y) \rightarrow (x + 2, y + 3)\)
Write the equation of the image of \( y = f(x) \) after each transformation.

a) A horizontal translation of 5 units left.  
   \[ (x, y) \rightarrow (x - 5, y) \]

b) A translation of 3 units up.  
   \[ (x, y) \rightarrow (x, y + 3) \]

c) A translation of \( m \) units right and \( p \) units down.  
\[ (x, y) \rightarrow (x + m, y - p) \]

d) \( (x, y) \rightarrow (x - 6, y + 1) \)

---

The graph of \( f(x) \) is shown. Sketch:

a) \( y = f(x - 2) \)

b) \( y + 2 = f(x + 3) \)

---

What happens to the graph of the function \( y = f(x) \) if the following changes are made to its equation?

a) replace \( x \) with \( x + 2 \)

b) replace \( y \) with \( y - 8 \)

---

**Assignment**

1. Describe how the graphs of the following functions relate to the graph of \( y = f(x) \).

\[ \begin{align*}
   \text{a)} & \quad y = f(x + 9) \\
   \text{b)} & \quad y = f(x) + 7 \\
   \text{c)} & \quad y = f(x - 4) + 4 \\
   \text{d)} & \quad y - 6 = f(x) \\
   \text{e)} & \quad y = 3 + f(x - 5) \\
   \text{f)} & \quad y + 2 = f(x + 3) - 10
\end{align*} \]
2. Write the equation of the image of \( y = f(x) \) after each transformation.
   a) A vertical translation of 10 units down.
   b) A horizontal translation of 8 units right and a vertical translation of 9 units up.
   c) A translation of \( t \) units up and \( s \) units left.

3. The function \( y = f(x) \) is transformed to \( y = f(x - h) + k \). Find the values of \( h \) and \( k \) for the following translations.
   a) 7 units right        b) 4 units up and 2 units left      c) \( a \) units right and \( b \) units down.

4. The point \((-3, 5)\) lies on the graph of \( y = f(x) \). State the coordinates of the image of this point under the following transformations.
   a) \( y = f(x) + 3 \)        b) \( y + 5 = f(x + 2) \)      c) \((x, y) \rightarrow (x - 7, y - 1)\)

5. Given the graph of the function \( y = f(x) \) sketch the graph of the indicated function.
   a) \( y = f(x - 4) \)
   b) \( y - 3 = f(x) \)
   c) \( y = f(x + 2) - 3 \)
   d) \( y + 2 = f(x - 5) \)
6. What happens to the graph of the function $y = f(x)$ if you make these changes to its equation?
   a) replace $x$ with $x - 8$
   b) replace $y$ with $y + 2$
   c) replace $x$ with $x + 4$ and $y$ with $y - 7$

7. The function $y = f(x)$ is transformed to $y = f(x + 2) + 4$. If the point $(3, -1)$ lies on the graph of $y = f(x)$, which of the following points must lie on the graph of $y = f(x + 2) + 4$?
   A. (5, 3)
   B. (1, 3)
   C. (7, 1)
   D. (7, -3)

8. The function $y = f(x)$ is transformed to $y - 3 = f(x - 1)$. If the point $(-2, 4)$ lies on the graph of $y - 3 = f(x - 1)$, which of the following points must lie on the graph of $y = f(x)$?
   A. $(-1, 7)$
   B. $(-1, 1)$
   C. $(-3, 7)$
   D. $(-3, 1)$

9. The graph of $y = g(x)$ was transformed to the graph of $y = g(x - 7) + 2$. Which of the following statements describes the transformation?
   A. The graph of $y = g(x)$ has been translated 2 units to the right and 7 units upward.
   B. The graph of $y = g(x)$ has been translated 7 units to the left and 2 units downward.
   C. The point $(x, y)$ on the graph $y = g(x)$ has been translated to point $(x + 7, y + 2)$.
   D. The point $(x, y)$ on the graph $y = g(x)$ has been translated to point $(x - 7, y - 2)$.

**Answer Key**

1. a) horizontal translation 9 units left
   b) vertical translation 7 units up
   c) translation 4 units right and 4 units up
   d) vertical translation 6 units up
   e) translation 5 units right and 3 units up
   f) translation 3 units left and 12 units down

2. a) $y = f(x - 10)$
   b) $y = f(x - 8) + 9$
   c) $y = f(x + s) + t$

3. a) $h = 7, k = 0$
   b) $h = -2, k = 4$
   c) $h = a, k = -b$

4. a) $(-3, 8)$
   b) $(-5, 0)$
   c) $(-10, 4)$

5. a) the graph is translated 4 units right
   b) the graph is translated 3 units up
   c) the graph is translated 2 units left and 3 units down
   d) the graph is translated 5 units right and 2 units down

6. a) horizontal translation 8 units right
   b) vertical translation 2 units down
   c) translation 4 units left and 7 units up

7. B
   8. D
   9. C

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Transformations Lesson #4: Horizontal and Vertical Translations Part 2

Warm-Up

In the previous lesson we had the following note:

Given the function \( y = f(x) \):

- replacing \( y \) with \( y = y - k \), \( (i.e. \ y \rightarrow y - k) \) describes a vertical translation.  
  \( y - k = f(x) \) or \( y = f(x) + k \) describes a vertical translation.

- replacing \( x \) with \( x = x - h \), \( (i.e. \ x \rightarrow x - h) \) describes a horizontal translation.  
  \( y = f(x - h) \) describes a horizontal translation.

In general, if \( y - k = f(x - h) \) or \( y = f(x - h) + k \) then

- \( k > 0 \) the graph moves up \( \uparrow \)
- \( k < 0 \) the graph moves down \( \downarrow \)
- \( h > 0 \) the graph moves right \( \rightarrow \)
- \( h < 0 \) the graph moves left \( \leftarrow \)

Class Ex. #1

Describe how the graph of the second function compares to the graph of the first function.

a) \( y = x^4 \)
\( y = x^4 + 3 \)

b) \( y = 6x - 3 \)
\( y = 6(x - 1) - 3 \)

c) \( y = |x| \)
\( y = |x - 6| + 2 \)

d) \( y = \frac{1}{\sqrt{x}} \)
\( y = \frac{1}{\sqrt{x + 1}} \)

Class Ex. #2

Write the equation of the image of:

a) \( y = x^2 \) after a horizontal translation of 3 units to the right.

b) \( y = 10^x \) after a vertical translation of 2 units up.

c) \( y = \sqrt{x} \) after a horizontal translation of 4 units to the left and a vertical translation of 3 units down.
The function represented by the thick line is a transformation of the function represented by the thin line. Write an equation for each function represented by the thick line.

a) 

\[ y = |x| \]

b) 

\[ y = \frac{1}{x} \]

c) 

\[ y = f(x) \]

d) 

\[ y = f(x) \]

\[ y = \sqrt{x} \] is a radical function.

a) What vertical translation would be applied to \( y = \sqrt{x} \) so that the translation image passes through (16, 7)?

b) What horizontal translation would be applied to \( y = \sqrt{x} \) so that the translation image passes through (17, 8)?
On a certain route trains travel at an average speed of 90km/h. The distance $d$, in kilometres, they travel can be described as a function of time, $t$, in hours, and represented by the equation $d = f(t) = 90t$.

A train leaves the station at 12:00 p.m. ($t = 0$). A second train travels with the same average speed, but leaves 3 hours later.

a) Write an equation which describes the distance travelled by the second train.

b) Sketch a distance time graph for each train on the grid.

| Complete Assignment Questions #1 - #8 |

**Assignment**

1. Describe how the graph of the second function compares to the graph of the first function.

   a) $y = x^3$
      
      $y = x^3 - 1$

   b) $y = 7x - 1$
      
      $y = 7(x - 3) - 1$

   c) $y = \cos x^\circ$
      
      $y = \cos (x + 45)^\circ$

   d) $y = |x|$
      
      $y + 3 = |x + 6|$

   e) $y = \frac{1}{x^2 + 1}$
      
      $y - 2 = \frac{1}{(x - 3)^2 + 1}$

   f) $y = a^x$
      
      $y = a^x + 1 + 1$

2. Write the equation of the image of:

   a) $y = x^4$ after a horizontal translation of 2 units to the left.

   b) $y = 2|x|$ after a translation of 3 units down and 1 unit left.

   c) $y = \frac{1}{\sqrt{x}}$ after a horizontal translation of 3 units to the right and a vertical translation of 2 units up.
3. The function represented by the thick line is a transformation of the function represented by the thin line. Write an equation for each function represented by the thick line.

a) \( y = x^2 \)

b) \( y = x^3 + 1 \)

c) \( y = \frac{1}{x - 3} \)

d) \( y = f(x) \)

e) \( y = f(x) \)

4. a) What vertical translation would be applied to \( y = x^2 \) so that the translation image passes through \((3, 5)\)?

b) What horizontal translation would be applied to \( y = x^3 + 1 \) so that the translation image passes through \((5, 28)\)?

c) What horizontal translation would be applied to \( y = \frac{1}{x - 3} \) so that the translation image passes through \((1, \frac{1}{2})\)?
5. On a certain route into town, shuttle buses depart every 15 minutes from 06:30 until 07:30. The distance \( d \), in kilometres, they travel can be described as a function of time, \( t \), in hours, and represented by the equation \( d = f(t) = 60t \).

If \( t = 0 \) at 06:30, write an equation which represents the distance travelled by:

a) the second bus  
b) the third bus  
c) the last bus

6. The graph of the function \( y = f(x) \) passes through the point (4, 7). Under a transformation, the point (4, 7) is transformed to (6, 6). A possible equation for the transformed function is

A. \( y - 1 = f(x + 2) \)  
B. \( y - 2 = f(x + 1) \)  
C. \( y + 1 = f(x - 2) \)  
D. \( y + 2 = f(x - 1) \)

7. The function \( f(x) = \sqrt{x} + 5 \) is transformed by a translation of 2 units down and 4 units to the left. The transformed function passes through the point (20, \( y \)).

To the nearest tenth, the value of \( y \) is _____ .

8. The function \( r(x) = \frac{1}{x + 3} \) is transformed by a translation of 3 units up and 5 units to the right. The transformed function passes through the point (\( x, 7 \)).

The value of \( x \) to the nearest hundredth is _____ .
Answer Key

1. a) vertical translation 1 unit down  
   b) translation 3 units right  
   c) horizontal translation 45° left  
   d) translation 6 units left and 3 units down  
   e) translation 3 units right and 2 units up  
   f) translation 1 unit left and 1 unit up

2. a) \( y = (x + 2)^4 \)  
   b) \( y = 2 \left| x + 1 \right| - 3 \)  
   c) \( y = \frac{1}{\sqrt{x - 3}} + 2 \)

3. a) \( y = \left| x - 3 \right| + 1 \)  
   b) \( y = (x - 6)^2 - 10 \)  
   c) \( y = \frac{1}{x - 3} + 1 \)  
   d) \( y = f(x) + 4 \)  
   e) \( y = f(x + 6) + 4 \)  
   f) \( y = f(x - 1) - 2 \)

4. a) vertical translation 4 units down  
   b) horizontal translation 2 units right  
   c) horizontal translation 4 units left

5. a) \( d = 60 \left( t - \frac{1}{4} \right) \)  
   b) \( d = 60 \left( t - \frac{1}{2} \right) \)  
   c) \( d = 60(t - 1) \)

6. C  
7. 7.9  
8. 2.25
Transformations Lesson #5: Reflections Part 1

**Invariant Points**

Invariant points are points on a graph which do not move after a transformation.

**Warm-Up #1** Comparing the Graphs of \( y = f(x) \) and \( y = -f(x) \)

**Part 1**

a) The graph of \( y = f(x) = x^2 - 10x + 25 \) is shown. Write an equation which represents \( y = -f(x) \).

b) Use a graphing calculator to sketch \( y = -f(x) \) and show the graph on the grid.

c) State the coordinates of the invariant point(s).

**Part 2**

a) The graph of \( y = f(x) = x^3 - 8 \) is shown. Write an equation which represents \( y = -f(x) \).

b) Use a graphing calculator to sketch \( y = -f(x) \) and show the graph on the grid.

c) State the coordinates of the invariant point(s).

d) How does the graph of \( y = -f(x) \) compare with the graph of \( y = f(x) \)?

**Part 3**

The graph of \( y = f(x) \) is shown. Sketch the graph of \( y = -f(x) \).
Warm-Up #2  Comparing the Graphs of $y = f(x)$ and $y = f(-x)$

Part 1

a) The graph of $y = f(x) = x^2 - 10x + 25$ is shown. Write an equation which represents $y = f(-x)$.

b) Use a graphing calculator to sketch $y = f(-x)$ and show the graph on the grid.

c) State the coordinates of the invariant point(s).

Part 2

a) The graph of $y = f(x) = x^3 - 8$ is shown. Write an equation which represents $y = f(-x)$.

b) Use a graphing calculator to sketch $y = f(-x)$ and show the graph on the grid.

c) State the coordinates of the invariant point(s).

d) How does the graph of $y = f(-x)$ compare with the graph of $y = f(x)$?

Part 3

The graph of $y = f(x)$ is shown. Sketch the graph of $y = f(-x)$.
**Warm-Up #3**

Comparing the Graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) or \( x = f(y) \)

**Part 1**

a) The graph of \( y = f(x) = (x - 5)^2 \) is shown. Write an equation which represents \( x = f(y) \) and solve for \( y \).

b) Use a graphing calculator to sketch \( x = f(y) \) and show the graph on the grid.

c) Although there are four points of intersection of the graphs, explain why there are only two invariant points. Mark the invariant points on the grid.

**Part 2**

a) The graph of \( y = f(x) = x^3 - 8 \) is shown. Write an equation which represents \( y = f^{-1}(x) \).

b) Use a graphing calculator to sketch \( y = f^{-1}(x) \) and show the graph on the grid.

c) Mark the invariant point(s) on the grid.

d) How does the graph of \( x = f(y) \) compare with the graph of \( y = f(x) \)?

**Part 3**

The graph of \( y = f(x) \) is shown. Sketch the graph of \( x = f(y) \).
Reflections

A reflection is a transformation which reflects (or flips) a figure about a line.

Fill in the following blanks which summarize Warm-Up #1 through to Warm-Up #3.

<table>
<thead>
<tr>
<th>Reflection</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection in the x-axis</td>
<td>If the graph of $y = f(x)$ is reflected in the ________, then it is the graph of ________ .</td>
<td><img src="image" alt="Graph of y = f(x)" /></td>
</tr>
<tr>
<td>Reflection in the y-axis</td>
<td>If the graph of $y = f(x)$ is reflected in the ________, then it is the graph of ________ .</td>
<td><img src="image" alt="Graph of y = f(x)" /></td>
</tr>
<tr>
<td>Reflection in the line $y = x$</td>
<td>If the graph of $y = f(x)$ is reflected in the line ________, then it is the graph of ________ or ________ .</td>
<td><img src="image" alt="Graph of y = f(x) and y = x" /></td>
</tr>
</tbody>
</table>

Given the function $y = f(x)$:

- replacing $x$ with $-x$, \( i.e. \ x \rightarrow -x \) describes a reflection in the y-axis. $y = f(-x)$ describes a reflection in the y-axis.

- replacing $y$ with $-y$, \( i.e. \ y \rightarrow -y \) describes a reflection in the x-axis. $-y = f(x)$ or $y = -f(x)$ describes a reflection in the x-axis.

- interchanging $x$ and $y$, \( i.e. \ x \rightarrow y, \ y \rightarrow x \) describes a reflection in the line $y = x$. $x = f(y)$ or $y = f^{-1}(x)$ describes a reflection in the line $y = x$. 

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The graph of $y = f(x)$ is shown. Sketch:

a) $y = -f(x)$  

b) $y = f(-x)$

c) $x = f(y)$

The graph drawn in the thick line is a reflection of the graph drawn in the thin line. Write an equation for each graph drawn in the thick line.

**Assignment**

1. The graph of $y = f(x)$ is shown. Sketch the graph of $y = -f(x)$.

a) 

b)
2. The graph of \( y = f(x) \) is shown. Sketch the graph of \( y = f(-x) \).

3. The graph of \( y = f(x) \) is shown. Sketch the graph of \( x = f(y) \).

4. The graph drawn in the thick line is a transformation of the graph drawn in the thin line. Write an equation for each graph drawn in the thick line and state whether this graph represents a function.
5. The function \( y = f(x) \) is transformed to the function below. Given that there are invariant points, describe the location of these points.

\[
\begin{align*}
\text{a)} & \quad y = -f(x) \\
\text{b)} & \quad y = f(-x) \\
\text{c)} & \quad x = f(y)
\end{align*}
\]

6. The point \((x, y)\) lies on the graph of the function \( y = f(x) \). State the coordinates of the image of \((x, y)\) under the following transformations:

\[
\begin{align*}
\text{a)} & \quad y = -f(x) \\
\text{b)} & \quad y = f(-x) \\
\text{c)} & \quad x = f(y)
\end{align*}
\]

7. Consider the graph of the function \( f(x) = x^2 \). Which of the following transformations would result in an identical graph?

A. \( -f(x) \)
B. \( f(-x) \)
C. \( -f(-x) \)
D. \( f(x + 1) \)

**Answer Key**

1. a) and b) graph is reflected in x-axis
2. a) and b) graph is reflected in y-axis
3. a) and b) graph is reflected in the line \( y = x \)
4. a) \( y = f(-x) \) is a function
b) \( x = f(y) \) is a not a function
c) \( y = -f(x) \) is a function
d) \( y = -f(-x) \) is a function
5. a) on the x-axis
b) on the y-axis
b) on the line \( y = x \)
6. a) \((x, -y)\)
b) \((-x, y)\)
c) \((y, x)\)
7. B
Transformations Lesson #6:
Reflections Part 2

Warm-Up

In the previous lesson we had the following note:

Given the function \( y = f(x) \):

- replacing \( x \) with \(-x\), \( i.e.\ x \rightarrow -x \) describes a reflection in the \( y \)-axis.
  \( y = f(-x) \) describes a reflection in the \( y \)-axis.

- replacing \( y \) with \(-y\), \( i.e.\ y \rightarrow -y \) describes a reflection in the \( x \)-axis.
  \(-y = f(x)\) or \( y = -f(x) \) describes a reflection in the \( x \)-axis.

- interchanging \( x \) and \( y \), \( i.e.\ x \rightarrow y,\ y \rightarrow x \) describes a reflection in the line \( y = x \)
  \( x = f(y)\) or \( y = f^{-1}(x) \) describes a reflection in the line \( y = x \).

Class Ex. #1

Write the equation of the image of:

a) \( y = x^2 \) after a reflection in the line \( y = x \)

b) \( y = 10^x \) after a reflection in the \( y \)-axis

c) \( y = \sqrt{x} \) after a reflection in the \( x \)-axis.

Class Ex. #2

Describe how the graph of the second function compares to the graph of the first function.

a) \( y = x^3 \)  \( y = -x^3 \)
b) \( y = 2^x \)  \( x = 2^y \)
c) \( y = \sin x \)  \( y = \sin(-x) \)

Class Ex. #3

The graph drawn in the thick line is a transformation of the graph drawn in the thin line. Write an equation for each graph drawn in the thick line and state whether this graph represents a function.

a) \( y = x \)

b) \( y = |x| \)
Class Ex. #4

a) Sketch the graph of \( f(x) = \frac{6}{x^2 + 3} \).

b) Write the equation for
   i) \( y = -f(x) \)   
   ii) \( y = f(-x) \)   
   iii) \( x = f(y) \)

c) Sketch each graph in b) and state whether the graph represents a function.
   i) \( y = -f(x) \)
   ii) \( y = f(-x) \)
   iii) \( x = f(y) \)

Class Ex. #5

a) Given \( f(x) = 3x + 2 \), determine:
   i) \( f^{-1}(x) \)
   ii) \( f(f^{-1}(x)) \)
   iii) \( f(f^{-1}(x)) \)

b) The graph of \( y = 3x + 2 \) is given. Sketch each graph in a);
   i) \( y = f^{-1}(x) \)
   ii) \( y = f^{-1}(f(x)) \)
   iii) \( y = f(f^{-1}(x)) \)

Note: The graphs of \( y = f^{-1}(f(x)) \) and \( y = f(f^{-1}(x)) \) will always be the line with equation \( y = x \) regardless of the function \( f \).

Complete Assignment Questions #1 - #8
Assignment

1. Write the equation of the image of:
   
   a) \( y = \frac{1}{x} \) after a reflection in the line \( y = x \)

   b) \( y = x^3 + x \) after a reflection in the \( y \)-axis

   c) \( y = |x| \) after a reflection in the \( x \)-axis.

   d) \( y = \sqrt{x - 2} \) after a reflection in the line \( y = x \)

   e) \( y = x^2 + 1 \) after a reflection in the \( y \)-axis

   f) \( y = \cos x \) after a reflection in the \( x \)-axis

2. Describe how the graph of the second function compares to the graph of the first function.
   
   a) \( y = 3x + 1 \) 
      \( y = -3x - 1 \)
   
   b) \( y = 3x + 1 \) 
      \( y = -3x + 1 \)
   
   c) \( y = 3x + 1 \) 
      \( x = 3y + 1 \)

   d) \( y = 10^x \) 
      \( y = 10^{-x} \)
   
   e) \( y = 10^x \) 
      \( y = -10^x \)
   
   f) \( y = 4x^2 \) 
      \( y = \pm \frac{\sqrt{x}}{2} \)
3. The graph drawn in the thick line is a transformation of the graph drawn in the thin line. Write an equation for each graph drawn in the thick line.

![Graphs](image1)

4.a) Sketch the graph of \( f(x) = (x - 1)^2 \).

b) Write the equation for:

i) \( y = -f(x) \)

ii) \( y = f(-x) \)

iii) \( x = f(y) \)

c) Sketch each graph in b) and state whether the graph represents a function.

i) \( y = -f(x) \)

ii) \( y = f(-x) \)

iii) \( x = f(y) \)
5.a) Sketch the graph the semi-circle of \( f(x) = \sqrt{16 - x^2} \).

b) Write the equation for:
   i) \( y = -f(x) \)
   ii) \( y = f(-x) \)
   iii) \( x = f(y) \)

c) Sketch each graph in b) and state whether the graph represents a function.
   i) \( y = -f(x) \)
   ii) \( y = f(-x) \)
   iii) \( x = f(y) \)

d) State the domain and range of each graph in c)

\[
\begin{array}{|c|c|c|}
\hline
& i) & ii) & iii) \\
\hline
\text{Domain} & & & \\
\hline
\text{Range} & & & \\
\hline
\end{array}
\]

Multiple Choice

6. The graph of \( y = 2x^5 \) is transformed to the graph of \( y = -2x^5 \). Consider the following statements about the transformed graph.

   i. It is a reflection of the original graph in the \( x \)-axis.
   ii. It is a reflection of the original graph in the \( y \)-axis.
   iii. It is a reflection of the original graph in the line \( y = x \).

How many of the above statements are false?

A. 0  
B. 1  
C. 2  
D. 3  

7. How could the graph of \( y = 2x^3 + 1 \) be used to graph \( y = -2x^3 + 1 \)?

A. Vertical translation of \( y = 2x^3 + 1 \).
B. Reflection of \( y = 2x^3 + 1 \) in the line \( y = x \).
C. Reflection of \( y = 2x^3 + 1 \) in the \( x \)-axis.
D. Reflection of \( y = 2x^3 + 1 \) in the \( y \)-axis.
The graphs below represent transformations of the graph of \( f(x) = x^2 \).

1. \( x = \frac{1}{y} \) or \( y = \frac{1}{x} \)
2. \( x = \sqrt{y - 2} \) or \( y = x^2 + 2, x \geq 0 \)
3. \( \frac{x}{3} + 2 \)
4. \( y = -\sqrt{x} \)

Match each of the graphs to the statement below.

- The graph represents \( f(-x) \)
- The graph represents \( f(x + 2) \)
- The graph represents \( f^{-1}(f(x)) \)
- The graph represents \( -f(x) \)

**Answer Key**

1.a) \( x = \frac{1}{y} \) or \( y = \frac{1}{x} \)  
   b) \( y = -x^3 - x \)  
   c) \( y = -|x| \)  
   d) \( x = \sqrt{y - 2} \) or \( y = x^2 + 2, x \geq 0 \)  
   e) \( y = x^2 + 1 \)  
   f) \( y = -\cos x \)

2.a) reflection in the \( x \)-axis  
   b) reflection in the \( y \)-axis  
   c) reflection in the line \( y = x \)  
   d) reflection in the \( y \)-axis  
   e) reflection in the \( x \)-axis  
   f) reflection in the line \( y = x \)

3.a) \( x = 3 - (y + 2)^2 \) or \( y = \pm \sqrt{3 - x - 2} \)  
   b) \( y = -(x + 3)(-x - 2)(-x - 4) \) or \( y = -(x - 3)(x + 2)(x + 4) \)  
   c) \( y = -\frac{x^2}{x + 3} \)  
   d) \( y = -\sqrt{x} \)

4.a) parabola opening up with vertex (1, 0)  
   b) \( y = -(x - 1)^2 \) or \( y = (x + 1)^2 \)  
   c) parabola opening down with vertex (1, 0). Is a function.  
   i) parabola opening up with vertex (1, 0). Is a function.  
   ii) parabola opening right with vertex (0, 1). Is not a function.

5.a) top half of a circle with centre at the origin and radius 4  
   b) \( y = -\sqrt{16 - x^2} \) or \( y = (x + 1)^2 \)  
   c) bottom half of a circle with centre at the origin and radius 4. Is a function.  
   ii) top half of a circle with centre at the origin and radius 4. Is a function.  
   iii) right half of a circle with centre at the origin and radius 4. Is not a function.

   d) \( y = -\sqrt{16 - y^2} \)

6. B  
7. D  
8. 3421
Transformations Lesson #7:  
Expansions and Compressions about the x- or y-axis Part 1

Warm-Up #1  
Comparing the Graphs of \( y = f(x) \) and \( y = af(x) \), where \( a > 0 \)

The graph of \( y = f(x) = \sqrt{4-x^2} \) is shown.

a) Write an equation which represents \( y = 3f(x) \).

b) Use a graphing calculator to sketch \( y = 3f(x) \) on the grid.

c) Describe how the number 3 in \( y = 3f(x) \) affects:
   - the general sketch of \( y = f(x) \) 
     (i.e. whether it expands or compresses \( y = f(x) \))
   - the \( x \)-intercepts of the graph of \( y = f(x) \)
   - the \( y \)-intercept of the graph of \( y = f(x) \).

d) Write an equation which represents \( y = \frac{1}{2}f(x) \).

e) Use a graphing calculator to sketch \( y = \frac{1}{2}f(x) \) on the grid.

f) Describe how the number \( \frac{1}{2} \) in \( y = \frac{1}{2}f(x) \) affects:
   - the general sketch of \( y = f(x) \) 
     (i.e. whether it expands or compresses \( y = f(x) \))
   - the \( x \)-intercepts of the graph of \( y = f(x) \)
   - the \( y \)-intercept of the graph of \( y = f(x) \).

g) Compared to the graph of \( y = f(x) \), the graph of \( y = af(x) \)
   results in a ___________ stretch about the ___-axis by a factor of \( a \).
   - If \( a > 1 \), the stretch is an ____________ .
   - If \( 0 < a < 1 \), the stretch is a ____________ .

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Warm-Up #2  Comparing the Graphs of \( y = f(x) \) and \( y = f(bx) \), where \( b > 0 \)

The graph of \( y = f(x) = \sqrt{4 - x^2} \) is shown.

a) Write an equation which represents \( y = f(4x) \).

b) Use a graphing calculator to sketch \( y = f(4x) \) on the grid.

c) Describe how the number 4 in \( y = f(4x) \) affects:
   - the general sketch of \( y = f(x) \) (i.e. whether it expands or compresses \( y = f(x) \))
   - the \( x \)-intercepts of the graph of \( y = f(x) \)
   - the \( y \)-intercept of the graph of \( y = f(x) \).

d) Write an equation which represents \( y = f \left( \frac{1}{3} x \right) \).

e) Use a graphing calculator to sketch \( y = f \left( \frac{1}{3} x \right) \) on the grid.

f) Describe how the number \( \frac{1}{3} \) in \( y = f \left( \frac{1}{3} x \right) \) affects:
   - the general sketch of \( y = f(x) \) (i.e. whether it expands or compresses \( y = f(x) \))
   - the \( x \)-intercepts of the graph of \( y = f(x) \)
   - the \( y \)-intercept of the graph of \( y = f(x) \).

g) Compared to the graph of \( y = f(x) \), the graph of \( y = f(bx) \) results in a __________ stretch about the ___-axis by a factor of \( \frac{1}{b} \).
   - If \( b > 1 \), the stretch is a ______________ .
   - If \( 0 < b < 1 \), the stretch is an ______________ .
Warm-Up #3  Comparing the Graphs of \( y = f(x) \) and \( y = af(x) \), where \( a < 0 \)

The graph of \( y = f(x) = x^3 - 2x^2 - 5x + 6 \) is shown.

a) Write an equation which represents \( y = -3f(x) \).

b) Use a graphing calculator to sketch \( y = -3f(x) \).

c) Describe how the number \(-3\) in \( y = -3f(x) \) affects:
   • the general sketch of \( y = f(x) \)
   • the \( x \)-intercepts of the graph of \( y = f(x) \)
   • the \( y \)-intercept of the graph of \( y = f(x) \).

d) Compared to the graph of \( y = f(x) \), the graph of \( y = af(x) \), where \( a < 0 \), results in a __________ stretch about the ___-axis by a factor of \( |a| \) together with a reflection in the _____-axis.
Warm-Up #4    Comparing the Graphs of \( y = f(x) \) and \( y = f(bx) \), where \( b < 0 \)

The graph of \( y = f(x) = x^3 - 2x^2 - 5x + 6 \) is shown.

\[ a) \text{ Write an equation which represents } y = f\left(\frac{-1}{3}x\right). \]

\[ b) \text{ Use a graphing calculator to sketch } y = f\left(\frac{-1}{3}x\right). \]

\[ c) \text{ Describe how the number } -\frac{1}{3} \text{ in } y = f\left(\frac{-1}{3}x\right) \text{ affects:} \]

\[ \text{• the general sketch of } y = f(x) \]

\[ \text{• the } x\text{-intercepts of the graph of } y = f(x) \]

\[ \text{• the } y\text{-intercept of the graph of } y = f(x). \]

\[ d) \text{ Compared to the graph of } y = f(x), \text{ the graph of } y = f(bx), \text{ where } b < 0, \]

results in a ___________ stretch about the ___-axis by a factor of \( \frac{1}{|b|} \) together with 
a reflection in the _____-axis.
### Expansions and Compressions

An expansion or a compression on a graph are transformations which stretch the graph vertically or horizontally.

The graph of \( y = f(x) \) and the graph of \( y = af(x) \) or \( y = f(bx) \) is given. Fill in the blanks in the table.

<table>
<thead>
<tr>
<th>( a ) or ( b )</th>
<th>Expansion or Compression</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a &lt; 0 )</td>
<td>The graph of ( y = f(x) ) will be reflected in the _______ and expanded or compressed vertically about the ___-axis.</td>
<td><img src="image1" alt="Graph" /></td>
</tr>
<tr>
<td>( 0 &lt; a &lt; 1 )</td>
<td>The graph of ( y = f(x) ) will be __________ __________ by a factor of ___ about the ___-axis.</td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>( a &gt; 1 )</td>
<td>The graph of ( y = f(x) ) will be __________ __________ by a factor of ___ about the ___-axis.</td>
<td><img src="image3" alt="Graph" /></td>
</tr>
<tr>
<td>( 0 &lt; b &lt; 1 )</td>
<td>The graph of ( y = f(x) ) will be __________ __________ by a factor of ___ about the ___-axis.</td>
<td><img src="image4" alt="Graph" /></td>
</tr>
<tr>
<td>( b &gt; 1 )</td>
<td>The graph of ( y = f(x) ) will be __________ __________ by a factor of ___ about the ___-axis.</td>
<td><img src="image5" alt="Graph" /></td>
</tr>
<tr>
<td>( b &lt; 0 )</td>
<td>The graph of ( y = f(x) ) will be reflected in the _______ and expanded or compressed horizontally about the ___-axis.</td>
<td><img src="image6" alt="Graph" /></td>
</tr>
</tbody>
</table>
Note

\( y = af(x) \) can be written as \( \frac{1}{a}y = f(x) \).

Given the function \( y = f(x) \):

- replacing \( x \) with \( bx \), \( (i.e. \ x \rightarrow bx) \) describes a horizontal stretch about the y-axis.
  i.e. \( y = f(bx) \) describes a horizontal stretch.

- replacing \( y \) with \( \frac{1}{a}y \), \( (i.e. \ y \rightarrow \frac{1}{a}y) \) describes a vertical stretch about the x-axis.
  i.e. \( \frac{1}{a}y = f(x) \) or \( y = af(x) \) describes a vertical stretch.

In general, if \( \frac{1}{a}y = f(bx) \) or \( y = af(bx) \), then for

<table>
<thead>
<tr>
<th>( a )</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a &gt; 1 )</td>
<td>there is a vertical expansion</td>
</tr>
<tr>
<td>( 0 &lt; a &lt; 1 )</td>
<td>there is a vertical compression</td>
</tr>
<tr>
<td>( a &lt; 0 )</td>
<td>there is also a reflection in the x-axis</td>
</tr>
<tr>
<td>( b &gt; 1 )</td>
<td>there is a horizontal compression</td>
</tr>
<tr>
<td>( 0 &lt; b &lt; 1 )</td>
<td>there is a horizontal expansion</td>
</tr>
<tr>
<td>( b &lt; 0 )</td>
<td>there is also a reflection in the y-axis</td>
</tr>
</tbody>
</table>

Class Ex. #1

Write the replacement for \( x \) or \( y \) and write the equation of the image of \( y = f(x) \) after each transformation.

a) A horizontal expansion by a factor of 6 about the y-axis.

b) A vertical compression by a factor of \( \frac{1}{5} \) about the x-axis.

c) A reflection in the x-axis and a vertical expansion about the x-axis by a factor of 3.

d) A horizontal compression about the y-axis by a factor of \( \frac{1}{2} \) and a vertical compression about the x-axis by a factor of \( \frac{1}{4} \).
How does the graph of $3y = f(x)$ compare with the graph of $y = f(x)$?

What happens to the graph of the function $y = f(x)$ if you make these changes?

a) Replace $x$ with $4x$.

b) Replace $y$ with $\frac{1}{3}y$.

c) Replace $y$ with $6y$ and $x$ with $\frac{1}{3}x$.

The graph of $y = f(x)$ is shown.

Sketch $y = f(-2x)$.

Complete Assignment Questions #1 - #7
Assignment

1. Write the replacement for $x$ or $y$ and write the equation of the image of $y = f(x)$ after each transformation.

   a) A horizontal expansion by a factor of 3 about the $y$-axis.

   b) A vertical expansion by a factor of 6 about the $x$-axis.

   c) A horizontal compression about the $y$-axis by a factor of $\frac{5}{7}$.

   d) A vertical compression about the $x$-axis by a factor of $\frac{2}{3}$.

   e) A reflection in the $y$-axis and a horizontal expansion by a factor of 3 about the $y$-axis.

   f) A reflection in the $x$-axis and a vertical compression by a factor of $\frac{3}{4}$ about the $x$-axis.

   g) A reflection in the $y$-axis and a horizontal compression about the $y$-axis by a factor of $\frac{3}{4}$.

   h) A horizontal expansion about the $y$-axis by a factor of 4 and a vertical expansion about the $x$-axis by a factor of 4.

   i) A horizontal compression about the $y$-axis by a factor of 0.5, a vertical expansion by a factor of 2 about the $x$-axis and a reflection in the $x$-axis.
2. The function \( y = f(x) \) is transformed to \( y = af(bx) \). Determine the values of \( a \) and \( b \) for:
   a) A horizontal compression by a factor of \( \frac{2}{3} \) about the \( y \)-axis.
   
   b) A vertical expansion about the \( x \)-axis by a factor of 5.
   
   c) A horizontal expansion about the \( y \)-axis by a factor of \( \frac{5}{2} \) and a reflection in the \( y \)-axis.
   
   d) A vertical compression about the \( x \)-axis by a factor of \( \frac{1}{3} \), a horizontal compression about the \( y \)-axis by a factor of \( \frac{1}{10} \) and a reflection in the \( y \)-axis.

3. Consider the function \( f(x) = x^2 \).
   a) Determine the equation of the image of the function if it is expanded vertically by a factor of 4 about the \( x \)-axis.
   
   b) Determine the equation of the image of the function if it is compressed horizontally by a factor of \( \frac{1}{2} \) about the \( y \)-axis.
   
   c) What do you notice?
   
   d) Give an example of a function where the stretches in a) and b) would not result in the same image.

4. a) What information about the graph of \( y = f(kx) \) does \( k \) provide?
   
   b) What information about the graph of \( ky = f(x) \) does \( k \) provide?
   
   c) What information about the graph of \( y - k = f(x) \) does \( k \) provide?
   
   d) What information about the graph of \( y = f(x - k) \) does \( k \) provide?
   
   e) What information about the graph of \( y = kf(x) \) does \( k \) provide?
5. The graph of \( y = f(x) \) is shown. In each case:
   i) sketch the graph of the transformed function
   ii) state the domain and range of the transformed function
   iii) state the coordinates of any invariant points.

a) \( y = f(2x) \)

b) \( y = -2f(x) \)

c) \( y = \frac{1}{2}f\left(\frac{1}{2}x\right) \)

d) \( y = f(2x) \)

e) \( y = -2f(x) \)

f) \( y = \frac{1}{2}f\left(\frac{1}{2}x\right) \)
6. What happens to the graph of the function \( y = f(x) \) if the following replacements are made?
   
a) Replace \( x \) with \( \frac{1}{3} x \).
   
b) Replace \( y \) with \( 4y \).
   
c) Replace \( y \) with \( -2y \) and \( x \) with \( 4x \).
   
d) Replace \( y \) with \( y - 4 \) and \( x \) with \( -\frac{1}{4} x \).

7. The graph of \( y = f(x) \) is compressed vertically by a factor of \( \frac{1}{2} \) about the \( x \)-axis, compressed horizontally by a factor of \( \frac{1}{4} \) about the \( y \)-axis, and reflected in the \( y \)-axis. If the equation of the image is written in the form \( y = af(bx) \), the value of \( a - b \), to the nearest tenth, is ______.

**Answer Key**

1. a) \( x \to \frac{1}{3} x, \quad y = f\left(\frac{1}{3} x\right) \)
   
b) \( y \to \frac{1}{6} y, \quad y = 6f(x) \)
   
c) \( x \to \frac{7}{5} x, \quad y = f\left(\frac{7}{5} x\right) \)
   
d) \( y \to \frac{3}{2} y, \quad y = \frac{2}{3} f(x) \)
   
e) \( x \to \frac{1}{3} x, \quad y = f\left(\frac{1}{3} x\right) \)
   
f) \( y \to -\frac{4}{3} y, \quad y = -\frac{3}{4} f(x) \)
   
g) \( x \to -\frac{4}{3} x, \quad y = f\left(-\frac{4}{3} x\right) \)
   
h) \( x \to \frac{1}{4} x \) and \( y \to \frac{1}{4} y, \quad y = 4f\left(\frac{1}{4} x\right) \)
   
i) \( x \to 2x \) and \( y \to -\frac{1}{2} y, \quad y = -2f(2x) \)

2. a) \( a = 1 \quad b = \frac{3}{2} \)
   
b) \( a = 5 \quad b = 1 \)
   
c) \( a = 1 \quad b = \frac{2}{5} \)
   
d) \( a = \frac{1}{3} \quad b = -10 \)

3. a) \( y = 4f(x) = 4x^2 \)
   
b) \( y = (2x)^2 = 4x^2 \)
   
c) both transformations result in the same image
   
d) many possible answers including \( f(x) = x, f(x) = x^3, f(x) = x^2 + 1 \), etc.
4. a) horizontal stretch about the \( y \)-axis by a factor of \( \frac{1}{k} \)
b) vertical stretch about the \( x \)-axis by a factor of \( \frac{1}{k} \)
c) vertical translation of \( k \) units: up if \( k > 0 \), down if \( k < 0 \)
d) horizontal translation of \( k \) units: right if \( k > 0 \), left if \( k < 0 \)

e) vertical stretch about the \( x \)-axis by a factor of \( k \)

5. a) b) c)

Domain:

\[ \{ x \mid 0 \leq x \leq 2, x \in \mathbb{R} \} \]
\[ \{ x \mid 0 \leq x \leq 4, x \in \mathbb{R} \} \]
\[ \{ x \mid 0 \leq x \leq 8, x \in \mathbb{R} \} \]

Range:

\[ \{ y \mid -2 \leq y \leq 2, y \in \mathbb{R} \} \]
\[ \{ y \mid -4 \leq y \leq 4, y \in \mathbb{R} \} \]
\[ \{ y \mid -1 \leq y \leq 1, y \in \mathbb{R} \} \]

Invariant Points: (0, 0) (0, 0), (2, 0), (4, 0) (0, 0)

d) e) f)

Domain:

\[ \{ x \mid -\frac{1}{2} \leq x \leq \frac{5}{2}, x \in \mathbb{R} \} \]
\[ \{ x \mid -1 \leq x \leq 3, x \in \mathbb{R} \} \]
\[ \{ x \mid -2 \leq x \leq 6, x \in \mathbb{R} \} \]

Range:

\[ \{ y \mid -3 \leq y \leq 3, y \in \mathbb{R} \} \]
\[ \{ y \mid -6 \leq y \leq 6, y \in \mathbb{R} \} \]
\[ \{ y \mid -\frac{3}{2} \leq y \leq \frac{3}{2}, y \in \mathbb{R} \} \]

Invariant Points: (0, 3) (-1, 0), (1, 0), (3, 0) none

6. a) horizontal expansion about the \( y \)-axis by a factor of 2
b) vertical compression about the \( x \)-axis by a factor of \( \frac{1}{4} \)
c) horizontal compression about the \( y \)-axis by a factor of \( \frac{1}{4} \), vertical compression about the \( x \)-axis by a factor of \( \frac{1}{2} \), and a reflection in the \( x \)-axis.
d) horizontal expansion about the \( y \)-axis by a factor of 4, a reflection in the \( y \)-axis, followed by a vertical translation of 4 units up.

7. 4.5
Transformations Lesson #8:
Expansions and Compressions about the x- or y-axis Part 2

Warm-Up

In the previous lesson we had the following note:
y = af(x) can be written as \( \frac{1}{a}y = f(x) \).

Given the function \( y = f(x) \):

- replacing \( x \) with \( bx \), \( (i.e. \ x \rightarrow bx) \) describes a horizontal stretch about the \( y \)-axis.
  i.e. \( y = f(bx) \) describes a horizontal stretch.

- replacing \( y \) with \( \frac{1}{a}y \), \( (i.e. \ y \rightarrow \frac{1}{a}y) \) describes a vertical stretch about the \( x \)-axis.
  i.e. \( \frac{1}{a}y = f(x) \) or \( y = af(x) \) describes a vertical stretch.

In general, if \( \frac{1}{a}y = f(bx) \) or \( y = af(bx) \), then for:
\[
\begin{align*}
a > 1 & \text{ there is a vertical expansion} \\
0 < a < 1 & \text{ there is a vertical compression} \\
a < 0 & \text{ there is also a reflection in the } x\text{-axis} \\
b > 1 & \text{ there is a horizontal compression} \\
0 < b < 1 & \text{ there is a horizontal expansion} \\
b < 0 & \text{ there is also a reflection in the } y\text{-axis}
\end{align*}
\]

Write the equation of the image of:

a) \( y = x^2 \) after a horizontal compression about the \( y \)-axis by a factor of \( \frac{3}{4} \).

b) \( y = \sqrt{x} - 3 \) after a horizontal expansion by a factor of 4 about the \( y \)-axis and a vertical expansion by a factor of 2 about the \( x \)-axis.

c) \( y = 3x + 7 \) after a vertical compression about the \( x \)-axis by a factor of \( \frac{1}{3} \) and a reflection in the \( x \)-axis.
Class Ex. #2

Describe how the graph of the second function compares to the graph of the first function.

\[ a) \quad y = f(x) \]
\[ y = f\left(\frac{1}{2}x\right) \]
\[ b) \quad y = 2^x \]
\[ y = 2^{3x} \]
\[ c) \quad y = |x| \]
\[ y = -2|x| \]
\[ d) \quad y = |x| \]
\[ y = |-2x| \]
\[ e) \quad y = |x| \]
\[ y = 2\left|\frac{1}{3}x\right| \]
\[ f) \quad y = x^3 \]
\[ 3y = x^3 \]

Class Ex. #3

The function represented by the thick line is a stretch of the function represented by the thin line. Write an equation for each function represented by the thick line.

\[ a) \quad y = \frac{6}{x^2 + 1} \]
\[ b) \quad y = f(x) \]
\[ c) \quad y = (x + 4)(x - 2)(x - 6) \]
A polynomial function has the equation $P(x) = (x - 4)(x + 3)(x + 6)$. Determine the zeros and the $y$-intercept if the following transformations are applied.

a) $y = -3P(x)$

b) $y = P\left(-\frac{1}{2}x\right)$

---

**Assignment**

1. Write the equation of the image of:

   a) $y = |x + 1|$ after a vertical compression about the $x$-axis by a factor of $\frac{7}{9}$.

   b) $y = 2^x$ after a horizontal expansion by a factor of 3 about the $y$-axis.

   c) $y = \sqrt{x - 2}$ after a vertical expansion about the $x$-axis by a factor of 4 and a reflection in the $x$-axis.

   d) $y = \sqrt{x - 2}$ after a horizontal expansion about the $y$-axis by a factor of 4 and a reflection in the $y$-axis.

   e) $y = \sin x^0$ after a horizontal compression about the $y$-axis by a factor of $\frac{3}{4}$ and a vertical compression about the $x$-axis by a factor of $\frac{1}{2}$.

   f) $y = 2x - 11$ after a horizontal compression about the $y$-axis by a factor of $\frac{1}{3}$ and a reflection in the $y$-axis.
g) \( y = \frac{1}{x+3} \) after a horizontal expansion about the y-axis by a factor of 3, a vertical expansion about the x-axis by a factor of 2 and a reflection in the y-axis.

h) \( y = \frac{1}{x} + 3 \) after a vertical compression about the x-axis by a factor of \( \frac{1}{2} \), a horizontal compression about the y-axis by a factor of \( \frac{1}{4} \), and a reflection in both the x-axis and y-axis.

2. Describe how the graph of the second function compares to the graph of the first function.

a) \( y = \sqrt{x} \)

b) \( y = x^4 \)

c) \( y = 5x + 10 \)

\[ y = \sqrt{\frac{1}{3}x} \]

\[ \frac{1}{4}y = x^4 \]

\[ y = 5\left(-\frac{1}{4}x\right) + 10 \]

d) \( y = \cos x^\circ \)

e) \( y = 5^x \)

\[ y = 3\cos 2x^\circ \]

\[ y = 5^{0.5x} \]

\[ y = 6x - x^2 \]

\[ y = 6(2x) - (2x)^2 \]

g) \( y = \frac{1}{x+1} \)

h) \( y = \frac{1}{x+1} \)

i) \( y = \frac{1}{x+1} \)

\[ y = \frac{2}{x+1} \]

\[ 3y = \frac{1}{x+1} \]

\[ y = \frac{4}{2x+1} \]
3. The function represented by the thick line is a stretch of the function represented by the thin line. Write an equation for each function represented by the thick line.

a) \( y = P(x) \)

b) \( y = P(x) \)

c) \( y = |x| - 2 \)

d) \( y = |x - 2| \)

e) \( y = x^2 \)

f) \( y = \frac{1}{8} (x + 8)(x - 4)(x - 12) \)
4. A polynomial function has the equation $P(x) = (x - 5)(x - 2)(x + 1)$. Determine the zeros and the $y$-intercept if the following transformations are applied.

a) $y = 4P(x)$

b) $y = P(4x)$

c) $y = -\frac{1}{4}P(x)$

d) $y = P\left(-\frac{1}{4}x\right)$

5. A polynomial function has the equation $P(x) = x(x - 2)^2$. Determine the zeros and the $y$-intercept if the following transformations are applied.

a) $y = 7P(x)$

b) $y = P\left(\frac{4}{5}x\right)$
6. How is the graph of \( \frac{1}{4}y = x^2 \) related to the graph of \( y = x^2 \)?

A. \( y = x^2 \) has been expanded horizontally about the y-axis by a factor of 4.
B. \( y = x^2 \) has been compressed horizontally about the y-axis by a factor of \( \frac{1}{4} \).
C. \( y = x^2 \) has been expanded vertically about the x-axis by a factor of 4.
D. \( y = x^2 \) has been compressed vertically about the x-axis by a factor of \( \frac{1}{4} \).

Answer Key

1. a) \( y = \frac{7}{9} |x + 1| \)  
   b) \( y = 2^\frac{1}{3}x \)  
   c) \( y = -4\sqrt{x - 2} \)  
   d) \( y = \sqrt{\frac{1}{4}x - 2} \)  
   e) \( y = \frac{1}{2} \sin \left( \frac{4}{3}x \right) \)  
   f) \( y = -6x - 11 \)  
   g) \( y = \frac{2}{3x + 3} \) or \( y = \frac{6}{x - 9} \)  
   h) \( y = \frac{1}{8x} - \frac{3}{2} \)

2. a) horizontal expansion about the y-axis by a factor of 3  
   b) vertical expansion about the x-axis by a factor of 4  
   c) horizontal expansion about the y-axis by a factor of 4 and a reflection in the y-axis  
   d) vertical expansion about the x-axis by a factor of 3 and a horizontal compression about the y-axis by a factor of \( \frac{1}{2} \)  
   e) horizontal expansion about the y-axis by a factor of 2  
   f) horizontal compression about the y-axis by a factor of \( \frac{1}{2} \)  
   g) vertical expansion about the x-axis by a factor of 2  
   h) vertical compression about the x-axis by a factor of \( \frac{1}{3} \)  
   i) vertical expansion about the x-axis by a factor of 4, horizontal compression about the y-axis by a factor of \( \frac{1}{2} \), and a reflection in the x-axis

3. a) \( y = 3P(x) \)  
   b) \( y = 2P(2x) \)  
   c) \( y = 3|x| - 6 \)  
   d) \( y = 3 \left| \frac{1}{2}x \right| - 6 \)  
   e) \( y = \frac{1}{4}x^2 \)  
   f) \( y = (x + 4)(x - 2)(x - 6) \)  
   g) \( y = -P\left( \frac{1}{3}x \right) \)  
   h) \( y = \frac{1}{3}f(-x) \)

4. a) zeros: \(-1, 2, 5\)  
   b) zeros: \(-\frac{1}{4}, \frac{1}{2}, \frac{5}{4}\)  
   c) zeros: \(-1, 2, 5\)  
   d) zeros: \(4, -8, -20\)  
   y-intercept: 40  
   y-intercept: 10  
   y-intercept: \(-\frac{5}{2}\)  
   y-intercept: 10

5. a) zeros: \(0, 2\) y-intercept = 0  
   b) zeros: \(0, \frac{3}{2}\) y-intercept = 0

6. C
Transformations Lesson #9: Combining Transformations Part One

**Combining Transformations**

In the previous lessons, we have learned the following rules:

Given the equation of a function \( y = f(x) \):

- replacing \( x \) with \( x - h \) or \( y \) with \( y - k \) results in a translation
- replacing \( x \) with \( -x \) or \( y \) with \( -y \) results in a reflection
- replacing \( x \) with \( bx \) or \( y \) with \( \frac{1}{a}y \) results in an expansion or compression.

In this lesson we are going to find out what happens when we combine different types of transformations.

**Warm-Up #1**  
Combining a Horizontal Expansion and a Vertical Expansion

The graph of \( y = f(x) \) is shown.

a) Sketch the image of the function after a horizontal expansion by a factor of 3 about the \( y \)-axis followed by a vertical expansion by a factor of 2 about the \( x \)-axis.

b) Sketch the image of the function after a vertical expansion by a factor of 2 about the \( x \)-axis followed by a horizontal expansion by a factor of 3 about the \( y \)-axis.

c) Does the order in which the two expansions are performed make a difference to the final graph?

**Warm-Up #2**  
Combining a Vertical Translation and a Horizontal Expansion

The graph of \( y = f(x) \) is shown.

a) Sketch the image of the function after a vertical translation of 3 units up followed by a horizontal expansion by a factor of 2 about the \( y \)-axis.

b) Sketch the image of the function after a horizontal expansion by a factor of 2 about the \( y \)-axis followed by a vertical translation of 3 units up.

c) Does the order in which the vertical translation and the horizontal expansion are performed make a difference to the final graph?
Warm-Up #3  Combining a Vertical Translation and a Vertical Expansion

The graph of $y = f(x)$ is shown.

a) Sketch the image of the function after a vertical translation of 2 units down followed by a vertical expansion by a factor of 3 about the $x$-axis.

b) Sketch the image of the function after a vertical expansion by a factor of 3 about the $x$-axis followed by a vertical translation of 2 units down.

c) Does the order in which the vertical translation and the vertical expansion are performed make a difference to the final graph?

Warm-Up #4  Combining a Horizontal Translation and a Horizontal Expansion

The graph of $y = f(x)$ is shown.

a) Sketch the image of the function after a horizontal translation of 1 units right followed by a horizontal expansion by a factor of 2 about the $y$-axis.

b) Sketch the image of the function after a horizontal expansion by a factor of 2 about the $y$-axis followed by a horizontal translation of 1 units right.

c) Does the order in which the horizontal translation and the horizontal expansion are performed make a difference to the final graph?
Order of Transformations

We have seen that when two transformations are applied to a graph, the order in which the transformations are performed may or may not make a difference to the final graph.

In general, the order **DOES NOT** matter when:
- two translations are combined
- two stretches are combined
- a translation and a stretch at right angles to one another are combined
- reflections and stretches are combined.

The order **DOES** matter when:
- a translation and a stretch in the same direction are combined.
- most reflections and translations are combined.

**Note**

Unless otherwise indicated, use the following order to describe how to transform from one graph to another.

1. Expansions and/or compressions
2. Reflections
3. Translations.

Class Ex. #1

Describe the series of transformations required to transform graph A to graph B.

Class Ex. #2

Describe the series of transformations required to transform:

a) graph A to graph B
b) graph A to graph C
c) graph B to graph C.

Complete Assignment Questions #1 - #2
Describe which transformations are applied to a graph of a function when the following changes are made to its equation. Does the order in which the transformations are performed affect the final graph?

a) Replace \(x\) with \(3x\) and \(y\) with \(y + 4\).

b) Replace \(x\) with \(\frac{2}{3}x\), \(y\) with \(-3y\), and \(x\) with \(x + 2\).

---

A graph of the parabola \(y = x^2 + 1\) is shown. The following transformations are applied to \(y = x^2 + 1\) in the order shown:

- a horizontal translation to the left 2 units
- a reflection in the \(x\)-axis
- a vertical compression about the \(x\)-axis by a factor of \(\frac{1}{4}\).
- a vertical translation down 3 units.

a) For each transformation:
   - graph the image on the grid
   - write the replacement for \(x\) or \(y\) and the current equation in the table.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Replacement for (x) or (y)</th>
<th>Current Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a horizontal translation left 2 units</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a reflection in the (x)-axis</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>a vertical compression by a factor of (\frac{1}{4})</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>a vertical translation down 3 units</td>
<td></td>
</tr>
</tbody>
</table>

b) Write the equation which represents the final position of the graph.

c) Verify using a graphing calculator.

---

Complete Assignment Questions #3 - #5


**Assignment**

1. Describe the series of transformations required to transform:
   a) graph A to graph B.
   
   b) graph A to graph C.
   
   c) graph A to graph D.

2. Describe the series of transformations required to transform:
   a) graph A to graph B.
   
   b) graph A to graph C.

3. Describe which transformations are applied to a graph of a function when the following changes are made to its equation. Does the order in which the transformations are performed affect the final graph?
   a) Replace $x$ with $x + 2$ and $y$ with $-y$.
   
   b) Replace $x$ with $4x$ and $y$ with $y - 7$.
   
   c) Replace $x$ with $\frac{1}{3}x$, $y$ with $-2y$, and $y$ with $y + 2$. 
d) Replace $x$ with $2x$, $y$ with $\frac{1}{4}y$, $x$ with $-x$, and $y$ with $y + 10$.

4. A graph of the parabola $y = x^2 + 1$ is shown. The following transformations are applied to $y = x^2 + 1$ in the order shown:

- a vertical translation down 3 units
- a reflection in the $x$-axis
- a vertical compression by a factor of $\frac{1}{4}$
- a horizontal translation to the left 2 units

a) For each transformation:

- graph the image on the grid
- write the replacement for $x$ or $y$ and the current equation in the table.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Replacement for $x$ or $y$</th>
<th>Current Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a vertical translation down 3 units</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>a reflection in the $x$-axis</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>a vertical compression by a factor of $\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>a horizontal translation left 2 units</td>
<td></td>
</tr>
</tbody>
</table>

b) Write the equation which represents the final position of the graph.

c) Verify using a graphing calculator.

d) Explain why the equation in this example is different from the equation in Class Ex. #4.
The following information refers to question #5

Five students were asked to perform a combination of two transformations on the graph shown.

**Student I:** A reflection in the \(x\)-axis and a translation 3 units left.

**Student II:** A reflection in the \(y\)-axis and a translation 3 units left.

**Student III:** A horizontal expansion by a factor of 2 and a reflection in the \(x\)-axis.

**Student IV:** A translation 2 units up and a vertical compression by a factor of \(\frac{1}{2}\).

**Student V:** A translation 2 units right and a vertical compression by a factor of \(\frac{1}{2}\).

5. For how many of the students does the order in which the transformations are performed affect the final graph?

A. One student

B. Two students

C. Three students

D. Four students
Answer Key

1. a) vertical compression by a factor of \( \frac{1}{2} \) about the x-axis, horizontal compression by a factor of \( \frac{1}{2} \) about the y-axis, then a translation of 1.5 units right, and 5 units up
   b) vertical expansion by a factor of 2 about the x-axis, a reflection in the x-axis, then a vertical translation 1 unit down
   c) vertical compression by a factor of \( \frac{1}{2} \) about the x-axis, horizontal compression by a factor of \( \frac{1}{3} \) about the y-axis, a reflection in the x-axis, then a translation 6 units right and 6 units down

2. a) reflection in the y-axis, and a translation 7 units down
   b) reflection in the x-axis, and then a translation 5 units right and 7 units up

3. a) horizontal translation 2 units left and a reflection in the x-axis; no
   b) horizontal expansion by a factor of \( \frac{1}{4} \) about the y-axis, and a vertical translation 7 units up; no
   c) horizontal expansion by a factor of 3 about the y-axis, vertical compression by a factor of \( \frac{1}{2} \) about the x-axis, reflection in the x-axis and a vertical translation 2 units down; yes
   d) horizontal compression by a factor of \( \frac{1}{2} \) about the y-axis, vertical expansion by a factor of 4 about the x-axis, a reflection in the y-axis and a vertical translation 10 units down; yes

4. a)

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Replacement for x or y</th>
<th>Current Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 a vertical translation down 3 units</td>
<td>replace y with y + 3</td>
<td>( y + 3 = x^2 + 1 ) ( y = x^2 - 2 )</td>
</tr>
<tr>
<td>2 a reflection in the x-axis</td>
<td>replace y with -y</td>
<td>( -y = x^2 - 2 ) ( y = -x^2 + 2 )</td>
</tr>
<tr>
<td>3 a vertical compression of a factor of ( \frac{1}{4} )</td>
<td>replace y with 4y</td>
<td>( 4y = -x^2 + 2 ) ( y = -\frac{1}{4}x^2 + \frac{1}{2} )</td>
</tr>
<tr>
<td>4 a horizontal translation left 2 units</td>
<td>replace x with x + 2</td>
<td>( y = -\frac{1}{4}(x + 2)^2 + \frac{1}{2} )</td>
</tr>
</tbody>
</table>

b) \( y = -\frac{1}{4}(x + 2)^2 + \frac{1}{2} \)

d) The order of the transformations is different. In particular, changing the order of the vertical compression and the vertical translation will result in a different equation.

5. B
Transformations Lesson #10: Combining Transformations Part 2

Equations Combining Two or More Transformations

To apply a combination of transformations, consider the following:

\[ y = af[b(x - h)] + k \]

where;

|a| is the vertical expansion or compression factor.
If a is negative, then there is also a reflection in the x-axis.

\[ \frac{1}{b} \]

is the horizontal expansion or compression factor.
If b is negative, then there is also a reflection in the y-axis.

h is the horizontal translation where;
• if h > 0 the translation is to the right.
• if h < 0 the translation is to the left.

k is the vertical translation where;
• if k > 0 the translation is k units up.
• if k < 0 the translation is k units down.

Note: When graphing a combination of transformations from an equation use the following order:

Step 1: Sketch the original function.
Step 2: Sketch any expansions and/or compressions.
Step 3: Sketch any reflections.
Step 4: Sketch any translations.

Warm-Up #1

The graph of \( y = f(x) \) is shown.

Consider the function defined by the equation

\[ y = 2f\left(\frac{1}{2}(x + 5)\right) - 8. \]

a) If the equation is written in the form \( y = af[b(x - h)] + k \), state the values of \( a, b, h, \) and \( k \).

b) Write the transformations associated with the parameters \( a, b, h, \) and \( k \).

c) Put these transformations in an order which can be used to sketch the graph of the function. Sketch the graph of the function.
Warm-Up #2

The original graph in Warm-Up #1 has equation \( y = x^2 \).

a) Write the equation for the transformed function \( y = 2f\left(\frac{1}{2}(x + 5)\right) - 8 \).

b) Graph the equation in a) on a calculator and verify the sketch in Warm-Up #1 c).

The graph of \( y = f(x) \) is shown.

Sketch the graph of \( y = -2f(x - 3) + 1 \).

Consider the function \( y = f(x) \). In each case determine:

- the replacements for \( x \) and \( y \) which would result in the following combinations of transformations
- the equation of the transformed function in the form \( y = af(b(x - h)) + k \)

a) a horizontal compression by a factor of \( \frac{1}{4} \) about the \( y \)-axis and a vertical translation of 5 units down.

b) a vertical compression by a factor of \( \frac{3}{5} \) about the \( x \)-axis, a reflection in the \( y \)-axis, and a horizontal translation 2 units left.
A function $G(x) = x^3$ is transformed into a new function $P(x)$. To form the new function $P(x)$, $G(x)$ is compressed vertically about the $x$-axis by a factor of 0.25, reflected in the $y$-axis, and translated 3 units to the right. Write the equation of the new function $P(x)$.

Given the graph of $y = f(x)$, sketch the graph of the transformed function $y = f\left(\frac{1}{2} x + 3\right) - 8$
(Hint: Rewrite this function in the form $y = af(b(x - h) + k)$)

The function $f(x) = \sqrt{x}$ has been transformed into the function $g(x) = -2\sqrt{3x - 12} + 5$. Complete the following statement.

The function $f(x)$ has been transformed to the function $g(x)$ by compressing horizontally about the $y$-axis by a factor of ___________, expanding vertically about the $x$-axis by a factor of ___________, reflecting in the ___________, translating ___________ units up and ___________ units horizontally to the ___________.

Complete Assignment Questions #1 - #11
Assignment

1. Describe how the graph of \( y = f(x) \) can be transformed to the graph of
   a) \( y = f[2(x - 1)] + 5 \)
   
   b) \( y = 2f(x + 4) - 5 \)
   
   c) \( y = f \left( \frac{1}{2}x + 6 \right) + 1 \)

2. Consider the function \( y = f(x) \). In each case determine:
   - the replacements for \( x \) and \( y \) which would result in the following combinations of transformations
   - the equation of the transformed function in the form \( y = af[b(x - h)] + k \)
   
   a) a horizontal expansion by a factor of 3 about the \( y \)-axis and a vertical translation of 6 units up.
   
   b) a reflection in the \( y \)-axis, a horizontal translation of 3 units right, and vertical translation of 5 units down.
   
   c) a horizontal compression by a factor \( \frac{2}{3} \) about the \( y \)-axis, a vertical compression by a factor of \( \frac{2}{5} \) about the \( x \)-axis, a reflection in the \( x \)-axis, and a vertical translation of 1 unit up.
3. Describe how the graph of the second function compares to the graph of the first function.
   a) \( y = x^4 \), \(-4y = (x - 2)^4\)

   b) \( y = |x| \), \( y = \left| \frac{1}{3}(x + 2) \right| \)

   c) \( y = \sqrt{x} \), \( y - 1 = 2\sqrt{4x - 8} \)

4. In each case the combination of transformations are applied in the order given to transform
   the graph of \( y = f(x) \) to the graph of \( y = af[b(x - h)] + k \).
   Determine the values of \( a, b, h, \) and \( k \).
   a) a horizontal compression by a factor of \( \frac{3}{5} \) about the \( y \)-axis and a reflection in the \( x \)-axis.

   b) a vertical compression by a factor of \( \frac{1}{3} \) about the \( x \)-axis and a reflection in the \( y \)-axis.

   c) a vertical expansion by a factor of 2 about the \( x \)-axis, then a translation 5 units to the left
      and 2 units up.

   d) a horizontal expansion by a factor of 4 about the \( y \)-axis, a vertical expansion by a factor
      of 2 about the \( x \)-axis, a reflection in the \( y \)-axis and then a translation of 10 units down.

   e) a translation of 6 units right, then a horizontal compression by a factor of \( \frac{1}{2} \)
      about the \( y \)-axis and a reflection in the \( x \)-axis.
5. The function \( f(x) = \sqrt{x} \) is transformed into the function \( g(x) \) by expanding horizontally by a factor of 6 about the y-axis, expanding vertically by a factor of 3 about the x-axis, reflecting in the x-axis, and translating 1 unit up and \( \frac{1}{2} \) unit to the right. Write the equation for \( g(x) \).

6. The graph of \( y = f(x) \) is shown. Sketch the graph of:
   a) \( y + 4 = -\frac{1}{2} f(x + 2) \)
   b) \( y = -4 f \left( \frac{1}{2} x + 1 \right) \)

7. The function \( f(x) = \sin x^\circ \) is transformed into the function \( g(x) \) by compressing horizontally by a factor of \( \frac{1}{4} \) about the y-axis, compressing vertically by a factor of \( \frac{2}{3} \) about the x-axis, reflecting in the y-axis, and translating 5 units down. Write the equation for \( g(x) \).
8. The function \( f(x) = \frac{1}{x} \) is transformed into the function \( g(x) \) by expanding horizontally by a factor of 2 about the y-axis, expanding vertically by a factor of 5 about the x-axis, and translating 3 unit to the left. Write the equation for \( g(x) \).

9. The graph of \( y = f(x) \) is the semi-circle centred at the origin. Which of the following shows the graph of \( y = f(x) \) and \( y = f(2x - 4) \)?

- A.
- B.
- C.
- D.

10. The graph of \( y = \frac{1}{x} \) is transformed to the graph of \( y = \frac{1}{5(x - 3)} + 4 \) by a series of transformations. One of these transformations is a vertical stretch about the x-axis. The scale factor of the vertical stretch, to the nearest tenth, is ______.
11. Jack and Jill are working with the graph of the function \( f(x) = x^2 \).

Jack expands the graph vertically by a factor of 4 about the \( x \)-axis, followed by a translation 2 units up.

Jill takes the graph of \( f(x) = x^2 \), and translates it 2 units up, followed by a vertical expansion by a factor of 4 about the \( x \)-axis. The images of the two graphs are identical except for a vertical separation of \( k \) units, \( k > 0 \).

The value of \( k \), to the nearest tenth, is _____.

**Answer Key**

1. a) horizontal compression by a factor of \( \frac{1}{2} \) about the \( y \)-axis, then a translation 1 unit right and 5 units up  
   b) vertical expansion by a factor of 2 about the \( x \)-axis, then a translation 4 units left and 5 units down  
   c) horizontal expansion by a factor of 2 about the \( y \)-axis, then a translation 12 units left and 1 unit up

2. a) replace \( x \) with \( \frac{1}{2}x \) and \( y \) with \( y - 6 \)  
   \[ y = f\left(\frac{1}{2}x\right) + 6 \]
   b) replace \( y \) with \( \frac{1}{2}y \), \( y \) with \( -y \) and \( x \) with \( -x \)  
   \[ y = -2f(-x) \]
   c) replace \( x \) with \( -x \), \( x \) with \( x - 3 \), and \( y \) with \( y + 5 \)  
   \[ y = f(-(x - 3)) - 5 \] or \( y = f(-x + 3) - 5 \)
   d) replace \( x \) with \( \frac{3}{2}x \), \( y \) with \( \frac{5}{2}y \), \( y \) with \( -y \), and \( y \) with \( y - 1 \)  
   \[ y = \frac{3}{2}f\left(\frac{3}{2}x\right) + 1 \]

3. a) vertical compression by a factor of \( \frac{1}{4} \) about the \( x \)-axis, reflection in the \( x \)-axis, and a horizontal translation 2 units right  
   b) horizontal expansion by a factor of 3 about the \( y \)-axis, then a horizontal translation 2 units left  
   c) vertical expansion by a factor of 2 about the \( x \)-axis, horizontal compression by a factor of \( \frac{1}{4} \) about the \( y \)-axis, then a translation 2 units right and 1 unit up

4. a) \( a = -1 \) \( b = \frac{5}{3} \) \( h = 0 \) \( k = 0 \)  
   b) \( a = \frac{1}{3} \) \( b = -1 \) \( h = 0 \) \( k = 0 \)  
   c) \( a = 2 \) \( b = 1 \) \( h = -5 \) \( k = 2 \)  
   d) \( a = 2 \) \( b = -\frac{1}{4} \) \( h = 0 \) \( k = -10 \)  
   e) \( a = -1 \) \( b = 2 \) \( h = 3 \) \( k = 0 \)

5. \( g(x) = -3\sqrt{\frac{1}{6}(x - \frac{1}{2})} + 1 \)

6. a)  
   b)  

7. \( g(x) = \frac{2}{3}\sin(-4x^2) - 5 \)

8. \( g(x) = \frac{10}{x + 3} \)

9. D

10. 0.2

11. 6.0
Reciprocal Transformations

The reciprocal function of \( f(x) \) is \( \frac{1}{f(x)} \). For example, if a function has the equation \( y = x^2 - 5 \), the reciprocal function has equation \( y = \frac{1}{x^2 - 5} \). A reciprocal transformation transforms the graph of \( y = f(x) \) to the graph of \( y = \frac{1}{f(x)} \).

Warm-Up #1  Exploring a Reciprocal Transformation

a) Consider the function \( f(x) \) with equation \( y = x + 3 \). Write the equation of the reciprocal function.

b) The graphs of \( y = f(x) \) and \( y = \frac{1}{f(x)} \) and a partial table of values are shown.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = f(x) )</th>
<th>( y = \frac{1}{f(x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6</td>
<td>-3</td>
<td>-( \frac{1}{3} )</td>
</tr>
<tr>
<td>-5</td>
<td>-2</td>
<td>-( \frac{1}{2} )</td>
</tr>
<tr>
<td>-4</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
<td>Undefined</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>( \frac{1}{6} )</td>
</tr>
</tbody>
</table>

Complete the following statements using the graphs and table of values.

- The \( y \)-intercept of \( f(x) \) is _____.
- The \( y \)-intercept of \( \frac{1}{f(x)} \) is _____.
- The \( x \)-intercept of \( f(x) \) is _____.
- The equation of the vertical asymptote of \( \frac{1}{f(x)} \) is \( x = _____ \).
- Using the table of values, the invariant points are ( , ) and ( , ).
- The horizontal asymptote of \( y = \frac{1}{f(x)} \) is \( y = _____ \).

c) Complete the following:

- When \( f(x) = 0 \), the graph of \( y = \frac{1}{f(x)} \) has a ______________ asymptote.
- As \( f(x) \) approaches \( \pm \infty \), (positive or negative infinity), the graph of \( y = \frac{1}{f(x)} \) approaches closer to the ______________ asymptote.
Warm-Up #2 Exploring a Reciprocal Transformation

a) Consider the function \( f(x) \) with equation \( y = x^2 - 4 \). Write the equation of the reciprocal function.

b) The graphs of \( y = f(x) \) and \( y = \frac{1}{f(x)} \) and a partial table of values are shown.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = f(x) )</th>
<th>( y = \frac{1}{f(x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>12</td>
<td>( \frac{1}{12} )</td>
</tr>
<tr>
<td>-3</td>
<td>5</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>Undefined</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
<td>( -\frac{1}{3} )</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
<td>( -\frac{1}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
<td>( -\frac{1}{3} )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>Undefined</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>( \frac{1}{12} )</td>
</tr>
</tbody>
</table>

Complete the following statements using the graphs and table of values.

- The \( y \)-intercept of \( f(x) \) is \( x = \ldots \).
- The \( y \)-intercept of \( \frac{1}{f(x)} \) is \( x = \ldots \).
- The \( x \)-intercepts of \( f(x) \) are \( x = \ldots \) and \( x = \ldots \).
- The equations of the vertical asymptotes of \( \frac{1}{f(x)} \) are \( x = \ldots \) and \( x = \ldots \).
- The horizontal asymptote of \( y = \frac{1}{f(x)} \) is \( y = \ldots \).

c) In Warm-Up #1 we were able to determine the invariant points of the graphs of \( f(x) \) and \( \frac{1}{f(x)} \) by using the table of values for \( y = \pm 1 \). Explain why we can use the lines \( y = \pm 1 \) to find the invariant points of the graphs of \( f(x) \) and \( \frac{1}{f(x)} \). Mark these points on the above sketch.

d) Complete the following:

- When \( f(x) = \ldots \), the graph of \( y = \frac{1}{f(x)} \) has vertical asymptotes.
- As \( f(x) \) approaches \ldots the graph of \( y = \frac{1}{f(x)} \) approaches closer to the horizontal asymptote with equation \( y = \ldots \).
Properties of Reciprocal Transformations

Complete the following using Warm-Ups #1 and #2.

1. • When \( f(x) = 0 \), the graph of \( y = \frac{1}{f(x)} \) may have a ____________ ____________.
   • When \( f(x) \) is positive, \( \frac{1}{f(x)} \) is ____________.
   • When \( f(x) \) is negative, \( \frac{1}{f(x)} \) is ____________.

2. • When \( f(x) = 1 \), \( \frac{1}{f(x)} = ____ \). When \( f(x) = -1 \), \( \frac{1}{f(x)} = ____ \).
   • The invariant points for a reciprocal transformation can be found where the lines \( y = \pm 1 \) intersect the graphs of \( f(x) \) and \( \frac{1}{f(x)} \).

3. • When \( f(x) \) increases over an interval, \( \frac{1}{f(x)} \) ____________ over the same interval.
   • When \( f(x) \) decreases over an interval, \( \frac{1}{f(x)} \) ____________ over the same interval.

4. • When \( f(x) \) approaches zero, \( \frac{1}{f(x)} \) approaches \( \pm \infty \) and the graph of \( \frac{1}{f(x)} \) approaches a ____________ asymptote.
   • When \( f(x) \) approaches \( \pm \infty \), \( \frac{1}{f(x)} \) approaches zero and the graph of \( \frac{1}{f(x)} \) approaches a ____________ asymptote.

• Remember: \( f^{-1}(x) \) does NOT mean \( \frac{1}{f(x)} \). \( f^{-1}(x) \) represents the inverse of the function \( f(x) \).
   • The above properties can be used as a general aid to sketch the reciprocal function of \( f(x) \).

• Given \( y = f(x) \),
  replacing \( y \) with \( \frac{1}{y} \), \( \text{(i.e. } y \rightarrow \frac{1}{y} \text{)} \) describes a reciprocal transformation.
  \( \frac{1}{y} = f(x) \) or \( y = \frac{1}{f(x)} \) describes a reciprocal transformation.
Use a graphing calculator to sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same grid (use Zoom Decimal). The lines with equations $y = 1$ and $y = -1$ have been provided as a guide.

a) $y = f(x) = x^3$

b) $y = f(x) = 2x(x - 1)(x + 1)$

The graph of $y = f(x)$ is given.

a) Sketch the graph of $y = \frac{1}{f(x)}$. The lines with equations $y = 1$ and $y = -1$ have been provided as a guide.

b) In each case, write the equations of the asymptotes of the graph of $y = \frac{1}{f(x)}$.

i) ii)
The graph of \( g(x) = \frac{1}{f(x)} \) is shown.

The maximum point of \( g(x) \) is at \((-3, 2)\) and the y-intercept of \( g(x) \) is \( \frac{1}{5} \).

a) Given that \( f(x) \) is quadratic function, sketch the graph of \( y = f(x) \) on the grid and state the coordinates of the minimum point.

b) \( f(x) \) can be written in the form \( y - k = a(x - h)^2 \). Determine the values of \( a, h, \) and \( k \).
Assignment

1. The graph of $y = f(x)$ is given.
   a) Sketch the graph of $y = \frac{1}{f(x)}$.
   b) In each case, write the equations of the asymptotes of the graph of $y = \frac{1}{f(x)}$.

i) $y = 1$
   $y = -1$
   ii) $y = 1$
   $y = -1$

iii) iv)
2. a) A function \( f(x) \) has equation \( y = x + 3 \). Write the equation of the reciprocal function.

\[ y = \frac{1}{f(x)} \]

b) Sketch the graph of \( y = f(x) \) and \( y = \frac{1}{f(x)} \).

c) State the coordinates of the invariant points of this reciprocal transformation.

3. The graph of \( g(x) = \frac{1}{f(x)} \), where \( f(x) \) is a linear function, is shown. The graph of \( y = g(x) \) has a \( y \)-intercept of 1 and a vertical asymptote with equation \( x = -1 \).

a) State the zero of the function \( f(x) \).

b) Sketch the graph of \( y = f(x) \).

c) If the function \( f(x) \) has equation \( y = ax + b \), determine the values of \( a \) and \( b \).

4. The function \( f(x) = (x - 2)^2 \) is transformed into the function \( g(x) \) by compressing horizontally by a factor of \( \frac{1}{5} \), translating 2 units up, and taking its reciprocal value. Write the equation for \( g(x) \).
5. Consider the following statements:

1. The invariant points of the graphs of \( y = f(x) \) and \( y = -f(x) \) are the intersection points of the line \( y = 0 \) (the x-axis) and the graphs of \( f(x) \) and \( -f(x) \).
2. The invariant points of the graphs of \( y = f(x) \) and \( y = f(-x) \) are the intersection points of the line \( x = 0 \) (the y-axis) and the graphs of \( f(x) \) and \( f(-x) \).
3. The invariant points of the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) or \( x = f(y) \) are the intersection points of the line \( y = x \) and the graphs of \( f^{-1}(x) \) or \( x = f(y) \).
4. The invariant points of the graphs of \( y = f(x) \) and \( y = \frac{1}{f(x)} \) are the intersection points of the lines \( y = \pm 1 \) and the graphs of \( f(x) \) and \( \frac{1}{f(x)} \).
5. A function \( f(x) \) is transformed to a function \( g(x) \). All points of intersection of the graphs of \( f(x) \) and \( g(x) \) are invariant points.

How many of the statements are false?

A. 0  B. 1  C. 2  D. more than 2

6. The graph of \( y = f(x) \) passes through the point (4, 6). The graph of \( y = \frac{1}{4f(x)} \) passes through the point (4, \( p \)). The value of \( p \) to the nearest hundredth is _____.
**Transformations Lesson #12:**

**Absolute Value Transformations**

Recall the definition of absolute value.

\[ |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases} \]

An absolute value transformation transforms the graph of \( y = f(x) \) to the graph of \( y = |f(x)| \).

**Warm-Up #1**

**Comparing the Graphs of \( y = f(x) \) and \( y = |f(x)| \)**

a) A function \( f(x) \) has equation \( y = x - 1 \). Write the equation for \( y = |f(x)| \).  

\[
\begin{array}{c|c|c}
\hline
x & y = f(x) & y = |f(x)| \\
\hline
-4 & & \\
-3 & & \\
-2 & & \\
-1 & & \\
0 & & \\
1 & & \\
2 & & \\
3 & & \\
4 & & \\
\hline
\end{array}
\]

b) Complete the table of values for \( y = f(x) \) and \( y = |f(x)| \).  

| \( x \) | \( y = f(x) \) | \( y = |f(x)| \) |
|--------|------------|-------------|
| -4     |            |             |
| -3     |            |             |
| -2     |            |             |
| -1     |            |             |
| 0      |            |             |
| 1      |            |             |
| 2      |            |             |
| 3      |            |             |
| 4      |            |             |

Warm-Up #2

**Exploring an Absolute Value Transformation using a Graphing Calculator**

a) The graph of the function \( f(x) \) with equation \( y = x^2 - 4 \) is shown. Write the equation for \( y = |f(x)| \).  

\[
\begin{array}{c|c}
\hline
x & y = f(x) \\
\hline
-3 & \\
-2 & \\
-1 & \\
0 & \\
1 & \\
2 & \\
3 & \\
\hline
\end{array}
\]

b) Using the observations in Warm-up #1, sketch the graph of \( y = |f(x)| \).  

c) Use a graphing calculator to sketch \( y = |f(x)| \) to check your answer in b).  

d) Do the observations in Warm-up #1 also apply in Warm-up #2?
In general, given the function \( y = f(x) \), the graph of \( y = |f(x)| \) has the following characteristics:

- When \( f(x) \geq 0 \), (i.e. the graph of \( y = f(x) \) is above the \( x \)-axis), the graph of \( y = |f(x)| \) is identical to the graph of \( y = f(x) \).
- When \( f(x) < 0 \), (i.e. the graph of \( y = f(x) \) is below the \( x \)-axis), the graph of \( y = |f(x)| \) is a reflection of the graph of \( y = f(x) \) in the \( x \)-axis.

Warm-Up #3  
Combining a Reciprocal Transformation and an Absolute Value Transformation

a) The graph of \( f(x) = 1 - x^2 \) is shown. Sketch the graph of \( y = |1 - x^2| \).

b) Without using a graphing calculator, sketch the graph of the reciprocal function \( y = \frac{1}{|1 - x^2|} \).

c) Use a graphing calculator to confirm your graphs in a) and b).

The graph of the function \( y = f(x) \) is shown. Sketch:

a) \( y = |f(x)| \)  

b) \( y = \frac{1}{|f(x)|} \)
The function $f(x) = \cos x^\circ$ is transformed into the function $g(x)$ by expanding horizontally by a factor of 6, expanding vertically by a factor of 3, reflecting in the $x$-axis, and then taking the absolute value. Write the equation for $g(x)$.

### Summary of Transformations

Given the function $y = f(x)$, we have the following:

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Replacement for $x$ or $y$</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical Translation</td>
<td>$y \rightarrow y - k$</td>
<td>$y - k = f(x)$ or $y = f(x) + k$</td>
</tr>
<tr>
<td>Horizontal Translation</td>
<td>$x \rightarrow x - h$</td>
<td>$y = f(x - h)$</td>
</tr>
<tr>
<td>Reflection in the $x$-axis</td>
<td>$y \rightarrow -y$</td>
<td>$-y = f(x)$ or $y = -f(x)$</td>
</tr>
<tr>
<td>Reflection in the $y$-axis</td>
<td>$x \rightarrow -x$</td>
<td>$y = f(-x)$</td>
</tr>
<tr>
<td>Reflection in the line $y = x$</td>
<td>$x \rightarrow y$, $y \rightarrow x$</td>
<td>$x = f(y)$ or $y = f^{-1}(x)$</td>
</tr>
<tr>
<td>Vertical Stretch</td>
<td>$y \rightarrow \frac{1}{a}y$</td>
<td>$\frac{1}{a}y = f(x)$ or $y = af(x)$</td>
</tr>
<tr>
<td>Horizontal Stretch</td>
<td>$x \rightarrow bx$</td>
<td>$y = f(bx)$</td>
</tr>
<tr>
<td>Reciprocal</td>
<td>$y \rightarrow \frac{1}{y}$</td>
<td>$\frac{1}{y} = f(x)$ or $y = \frac{1}{f(x)}$</td>
</tr>
<tr>
<td>Absolute Value</td>
<td>$f(x) \rightarrow</td>
<td>f(x)</td>
</tr>
</tbody>
</table>

### Complete Assignment Questions #1 - #8
Assignment

1. The graph of \( f(x) \) is shown. Sketch \( y = |f(x)| \)
   a) 

   ![Graph of \( f(x) \) and \( y = |f(x)| \)]

   b) 

   ![Graph of \( y = |f(x)| \)]

2. The function \( f(x) = x^3 \) is transformed into the function \( g(x) \) by expanding vertically by a factor of 8, reflecting in the \( x \)-axis, and taking the absolute value. Write the equation for \( g(x) \).

3. Describe how the graph of the second function compares to the graph of the first function.
   a) \( y = \sqrt{x} \)
   b) \( y = x^2 \)
   c) \( y = x \)
   d) \( y = \frac{1}{\sqrt{2x}} \)
   e) \( y = \frac{1}{|x^2 - 9|} \)
   f) \( y = 4|x + 1| \)
4. The graph of \( y = f(x) \) is shown. Sketch \( y = \frac{1}{|f(x)|} \) on the grid provided.

\begin{align*}
a) & \quad \begin{array}{c}
\includegraphics[width=0.4\textwidth]{graph_a.png}
\end{array} \\
& \quad \begin{array}{c}
\includegraphics[width=0.4\textwidth]{graph_b.png}
\end{array}
\end{align*}

5. The function \( f(x) = x^3 \) is transformed into the function \( g(x) \) by compressing vertically by a factor of \( \frac{1}{2} \), translating 3 units left and 4 units down, and then taking the absolute value. Write the equation for \( g(x) \).
Use the following graphs to answer questions #6, #7, and #8

The graph of the function \( y = f(x) \) is shown.

Transformations on the graph \( y = f(x) \) are shown below.

1.

2.

3.

4.

5.

6.
6. The graph which represents \( y = \frac{1}{f(x)} \)

A. 1  
B. 2  
C. 3  
D. 4

7. The three graphs which represent \( y = |f(x)| \), \( y = |f(-x)| \) and \( y = -|f(x)| \) are respectively:

A. 2, 5, 6  
B. 2, 6, 5  
C. 6, 5, 2  
D. 6, 2, 5

8. The graph which represents \( y = \frac{1}{|f(x)|} \) is

A. 1  
B. 3  
C. 4  
D. 5

**Answer Key**

1. a) see graph below  
   ![Graph](image1.png)  
   b) see graph below  
   ![Graph](image2.png)

2. \( g(x) = | -8x^3 | \)

3. a) horizontal compression by a factor of \( \frac{1}{2} \) about the \( y \)-axis and a reciprocal transformation  
   b) vertical translation 9 units down, absolute value transformation, and a reciprocal transformation  
   c) horizontal translation 1 unit left, absolute value transformation, and vertical expansion by a factor of 4 about the \( x \)-axis.

4. a)  
   ![Graph](image3.png)  
   b)  
   ![Graph](image4.png)

5. \( y = \left| \frac{1}{2}(x + 3)^3 - 4 \right| \)

6. C  
7. B  
8. C
Exponential and Logarithmic Functions Lesson #1: Review of Exponents

Warm-Up Review of Exponent Laws

The parts of the power are listed below

\[ x^n \]

power

exponent

base

Complete the following:

**Product Law**  \( x^m x^n = \)

**Quotient Law**  \( x^m \div x^n = \)

**Power of a Power**  \( (x^m)^n = \)

**Power of a Product**  \( (xy)^m = \)

**Power of a Quotient**  \( \left( \frac{x}{y} \right)^m = \), \( y \neq 0 \)

**Integral Exponent Rule**  \( x^{-m} = \), \( x \neq 0 \)

**Rational Exponents**  \( x^n = \text{or} \left( \right)^m \)

Write each expression without brackets and with positive exponents.

a) \( \frac{1}{2} y^{-6} \)

b) \( \frac{5x^{-3}}{y^{-2}} \)

c) \( (4x^3y)(2x^{-4}y^2) \)

d) \( \frac{24m^5p^{-3}q^4}{-4m^4p^2q^2} \)

e) \( (3x^2y^3)^3 \)

f) \( \frac{12b^{\frac{-2}{3}}}{3b} \)
Find the exact value of the following. Verify with a calculator.

a) \(9^{-3}\)  

b) \(81^{\frac{3}{4}}\)  

c) \(25^0\)  

d) \(16^{-\frac{1}{2}}\)

Simplify the expression \(\frac{p^{-1}q^2}{4p^2r^2}\) and evaluate when \(p = 2\), \(q = 3\), and \(r = -5\).

Simplify the following. Write the answers with positive exponents.

a) \((4xy^{-2})^{-3}\)  

b) \(-a^5(b^{-3})^2\)  

c) \(\left(\frac{3x^3}{4y^{-2}}\right)^{-2}\)  

d) \(\left(-8x^8y^5\right)\left(\frac{5x^3y^9}{25x^2y}\right)^2\)
Changing Base

Convert each of the following to the base indicated.

a) \(9^{2x}\) to base 3  

b) \(125^{2-x}\) to base 5  

c) \(8 \cdot 16^x\) to base 2

\[ \frac{1}{512^{3x}} \text{ to base 2} \]

\[ \left(\frac{16}{81}\right)^{x+5} \text{ to base } \frac{2}{3} \]

Complete Assignment Questions #1 - #10

Assignment

1. Write each expression without brackets and with positive exponents.

   a) \(4x y^{-3}\)
   
   b) \(\frac{15y^{-3}}{5y}\)
   
   c) \((3x^3y)(5x^{-2}y^4)\)

   d) \(\frac{24p^{-8}}{16p^{-3}}\)

   e) \(\frac{2}{a^{-5}}\)

   f) \((2x^{-2})^3\)
2. Find the exact value of the following:
   a) \(5^{-2}\)  
   b) \(27^{\frac{4}{3}}\)  
   c) \(\left(\frac{4}{9}\right)^{-\frac{3}{2}}\)  
   d) \(125^{\frac{1}{3}} - 10^0(64)^{\frac{2}{3}}\)  
   e) \(\left(\frac{1}{4}\right)^{-2}\)

3. Evaluate the following expressions for \(a = 1\), \(b = -2\), and \(c = 3\).
   a) \((a^{-2}b^{-4})(a^2b^{-5})\)
   b) \(\frac{a^{-1}b^3c^{-2}}{abc}\)
   c) \(\frac{a^{-1} + b^{-1}}{c^{-1} + c^{-2}}\)

4. Simplify the following. Write the answers with positive exponents.
   a) \(\frac{x^5y^{-1}}{x^2y^{-4}}\)
   b) \(\frac{-a^{-2}(b^{-1})^2}{b^3(-a^{-4})^2}\)
   c) \(\left(\frac{5x^3}{2y^4}\right)^{-3}\)
   d) \((4m^2n)^{-1} \times 2mn^5\)
   e) \(\frac{-8x^8y^5}{24x^2y} \left(\frac{15x^3y^9}{18xy^5}\right)^2\)
   f) \(\frac{3x^2y^0z^{-4}}{(2xyz)^3}\)
5. Convert each of the following to the base indicated.
   a) \(32^x\) to base 2
   b) \(81^{x-2}\) to base 3
   c) \(\frac{1}{64^{2x}}\) to base 4
   d) \(\left(\frac{1}{16}\right)^{x+1}\) to base 2
   e) \(\left(\frac{25}{49}\right)^{3x}\) to base \(\frac{5}{7}\)
   f) \(\left(\frac{27}{64}\right)^{x+2}\) to base \(\frac{4}{3}\)

6. Convert each of the following to the base indicated.
   a) \(2 \cdot 4^x\) to base 2
   b) \(9 \cdot 27^{x-1}\) to base 3
   c) \(\frac{1}{4} \cdot \left(\frac{1}{16}\right)^{4-x}\) to base 4

Multiple Choice

7. \((4x^{-3}y^5)^2\) is equal to
   A. \(\frac{16y^{10}}{x^6}\)
   B. \(\frac{4y^{10}}{x^6}\)
   C. \(\frac{16y^{10}}{x^3}\)
   D. \(\frac{16x^6}{y^{10}}\)

8. \(\frac{(2a^2b)^{-3}}{(ab^2)^{-4}}\) is equal to
   A. \(\frac{b^5}{2a^2}\)
   B. \(\frac{b^5}{8a^2}\)
   C. \(\frac{8a^2}{b^5}\)
   D. \(\left(\frac{2a}{b}\right)^{-7}\)
9. \((36x^{-4})^{-\frac{1}{2}}\) is equal to

A. \(\frac{6}{x^2}\)
B. \(-18x^2\)
C. \(\frac{x^2}{6}\)
D. \(\frac{x^{-4.5}}{6}\)

10. \((64p^2q^{-2})^{-\frac{1}{2}}\) is equal to
\((p^5q^{10})^{-\frac{1}{2}}\)

A. \(\frac{8}{p^3}\)
B. \(-\frac{32}{q^4}\)
C. \(\frac{q^{\frac{7}{3}}}{8p}\)
D. \(\frac{q^{\frac{7}{3}}}{8}\)

**Answer Key**

1. a) \(\frac{4x}{y^3}\) b) \(\frac{3}{y^4}\) c) \(15xy^5\) d) \(\frac{3}{2p^3}\) e) \(2a^{\frac{1}{3}}\) f) \(\frac{8}{x^6}\)

2. a) \(\frac{1}{25}\) b) 81 c) \(\frac{27}{8}\) d) -11 e) 16

3. a) \(-\frac{1}{512}\) b) \(\frac{4}{27}\) c) \(\frac{9}{8}\)

4. a) \(x^3y^3\) b) \(\frac{-1}{a^{10}b^5}\) c) \(\frac{8y^{12}}{125x^9}\) d) \(\frac{n^4}{2m}\) e) \(\frac{-25x^{10}y^{12}}{108}\) f) \(\frac{3}{8xy^{3}z^{7}}\)

5. a) \(2^{5x}\) b) \(3^{4x-8}\) c) \(4^{-6x}\) d) \(2^{-4x-4}\) e) \(\left(\frac{5}{7}\right)^{6x}\) f) \(\left(\frac{4}{3}\right)^{-3x-6}\)

6. a) \(2^{2x+1}\) b) \(3^{3x-1}\) c) \(4^{2x-9}\)

Exponential and Logarithmic Functions Lesson #2: Solving Equations Involving Exponents

Warm-Up #1 Review

Simplify \((9^{2x+3} ÷ 27^{3x-1}) \times 81^{x-1}\) by converting each term to a common base.

Solving Equations with Rational Exponents

Use the following procedure to solve an equation where the exponent is rational:
- Raise both sides to the reciprocal power of the exponent
- Simplify and solve for the variable.

Class Ex. #1
Solve for \(x\) in the following.

a) \(x^{-\frac{4}{3}} = 81\)  
b) \((3x - 5)^{\frac{3}{2}} = 27\)

Solving Exponential Equations with a Common Base

An exponential equation is an equation where the variable is in the exponent.

Use the following procedure to solve an equation where the variable is in the exponent:
- Write both sides of the equation in the same base
- Equate the exponents on both sides of the equation
- Determine the value of the variable.

Class Ex. #2
Solve for \(x\) in

a) \(5^{2x + 3} = 5^7\)  
b) \(7^{x - 2} = 343\)  
c) \(3^{5x - 1} = 81^{3x}\)  
d) \(3^x = 27\sqrt{3}\)
Assignment

1. Simplify.
   a) $49^{x-1} \times 7^{2x-3}$
   b) $216^x + (1296^{5x-4} \times 36^{x+5})$
   c) $64^{3x} \times 128^{x-1} + (32^{2x+3} + 8^{4x-1})$

2. Solve for $x$.
   a) $x^{\frac{1}{2}} = 5$
   b) $x^{-\frac{1}{2}} = 5$
   c) $x^{\frac{1}{3}} = -5$
   d) $4x^{-\frac{2}{3}} = 16$

3. Solve for $x$.
   a) $2^x = 16\sqrt{2}$
   b) $2^{-x} = 64$
   c) $9^{3x+1} = 27^{3x}$
4. Solve for \( x \).

a) \( \left( \frac{4}{7} \right)^{5x} = \left( \frac{64}{343} \right)^{2x - 1} \)

b) \( \left( \frac{125}{216} \right)^{\frac{x}{2}} = \left( \frac{6}{5} \right)^{3x + 2} \)

c) \( 49 \left( \frac{7}{12} \right)^{2x} = 144 \)

d) \( \left( \frac{9}{4} \right)^{x + 3} = \left( \frac{8}{27} \right)^{-5} \)

e) \( 2^{x - 1} = (128^5)(2^5) \)

f) \( 2(6^{2y}) - 74(6^y) + 72 = 0 \)

(Hint: Write as a quadratic equation with the variable as \( 6^y \))
g) \( \left( \frac{1}{4} \right)^x = 12 \) \( \Rightarrow \) (2)(32)\(^2x + 1\)  

h) \( \sqrt[3]{\frac{272x - 1}{3x + 1}} = 9 \)

5. If \( 4^{2x - 7} = \frac{1}{64} \), then the value of \( \sqrt{x} \) is
   A. \( 2 \)
   B. \( \sqrt{2} \)
   C. \( \sqrt{5} \)
   D. \( \frac{3}{2} \)

6. The solution to the equation \( 25^{x + 1} = 5^{3(x - 1)} \) to the nearest tenth, is \( x = \ldots \).

7. The solution to the equation \( \left( \frac{1}{8} \right)^{x - 3} = (2)(16)^{2x + 1}, \) to the nearest hundredth, is \( x = \ldots \).

8. The solution to the equation \( 8^{2x - 1} = 16 \), to the nearest tenth, is \( x = \ldots \).

**Answer Key**

1. a) \( 7^{4x - 5} \)  
   b) \( 6^{-19x + 6} \)  
   c) \( 2^{27x - 25} \)

2. a) \( x = 25 \)  
   b) \( x = \frac{1}{25} \)  
   c) \( x = -125 \)  
   d) \( x = \frac{1}{8} \)

3. a) \( x = \frac{9}{2} \)  
   b) \( x = -6 \)  
   c) \( x = \frac{2}{3} \)

4. a) \( x = 3 \)  
   b) \( x = -\frac{4}{3} \)  
   c) \( x = -1 \)  
   d) \( x = \frac{9}{2} \)  
   e) \( x = -\frac{1}{7} \)  
   f) \( x = 0, x = 2 \)

   g) \( x = \frac{3}{2} \)  
   h) \( x = 2 \)  

5. B  
6. 5.0  
7. 0.36  
8. 1.2
Exponential and Logarithmic Functions Lesson #3: The Exponential Function

Warm-Up #1 Exponential Growth

Mallory, a medical research scientist, discovered a new bacteria culture which could help strengthen a person’s immune system. To find the growth rate, she isolated 5 cells of the culture and observed the following growth pattern:

- after one hour there were 10 cells
- after two hours there were 20 cells
- after three hours there were 40 cells

<table>
<thead>
<tr>
<th>t</th>
<th>N(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
</tr>
</tbody>
</table>

a) Let \( t \) represent the time in hours and \( N(t) \) represent the number of cells after \( t \) hours. The formula for \( N(t) \) as a function of \( t \) can be written in the form \( N(t) = ab^t \), where \( a \) and \( b \) are constants and \( a,b > 0 \). Determine the values of \( a \) and \( b \) and write the function.

b) Use the formula to determine how many cells there were after 8 hours.

c) After 12 hours there are 20 480 cells. How many hours did it take to have half that amount?

d) Use a graphing calculator to graph the function. A graph of this type represents exponential growth.

Warm-Up #2 Exponential Decay

Chernobyl is a city in the former Soviet Union. In April 1986, there was a nuclear accident and the atmosphere was contaminated with quantities of a type of radioactive iodine. At the time of the explosion the atmosphere was contaminated with 32 768 units of radioactive iodine. The following data was recorded for the dissipation of the radioactive iodine:

- after 1 week there was 16 384 units left in the atmosphere
- after 2 weeks there was 8192 units left in the atmosphere
- after 3 weeks there was 4096 units left in the atmosphere

a) Use \( t \) to represent the time in weeks and \( N(t) \) to represent the number of units of radioactive iodine after \( t \) weeks. Complete the following equation of the function which represents the information provided.

\[ N(t) = 32\,768 \left( \frac{1}{2} \right)^t \]

b) Use the formula to determine how many units of radioactive iodine were left after 14 weeks.

c) Use a graphing calculator to graph the function using the Window shown. A graph of this type represents exponential decay.
**Exponential Function**

Warm-Up #1 and Warm-Up #2 are examples of exponential functions. An exponential function is a function whose equation is of the form

\[ y = ab^x \]

where \( a \neq 0, b > 0, x \in \mathbb{R} \)

**Warm-Up #3**

Comparing the Graphs \( y = 2^x \) and \( y = \left(\frac{1}{2}\right)^x \)

a) State the values of \( a \) and \( b \) for

\( y = 2^x \) and \( y = \left(\frac{1}{2}\right)^x \)

b) Sketch the graph of the exponential function with equation \( y = 2^x, x \in \mathbb{R}, \) using the table of values and grid.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

d) Complete the following chart

<table>
<thead>
<tr>
<th>Equation of Function</th>
<th>Domain of Function</th>
<th>Range of Function</th>
<th>( x )-intercept of Graph</th>
<th>( y )-intercept of Graph</th>
<th>Equation(s) of Asymptotes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 2^x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \left(\frac{1}{2}\right)^x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

e) Complete the following statements using the words “growth” or “decay”.

- \( f(x) = 2^x \) is an example of a ________ function.
- \( f(x) = \left(\frac{1}{2}\right)^x \) is an example of a ________ function.
Warm-Up #4 Exploring the Value of $b$ in $y = ab^x$ where $a = 1$

a) By using a graphing calculator or other technology, sketch the exponential functions with equation:

(i) $y = 3^x$

(ii) $y = 10^x$

(iii) $y = \left(\frac{1}{3}\right)^x$

(iv) $y = \left(\frac{1}{10}\right)^x$

b) The value of $b$ affects the steepness of the graph as $x$ increases. Complete the following.

- When $b > 1$, the curve __________ more sharply as $b$ increases.
- When $0 < b < 1$, the curve __________ more sharply as $b$ decreases.

c) Without using a graphing calculator, make a sketch of the graphs of:

i) $y = 5^x$

ii) $y = (0.2)^x$

d) Verify the solution in c) using a graphing calculator

e) State the $x$-intercept for each of the graphs of the form $y = b^x$.

f) State the $y$-intercept for each of the graphs of the form $y = b^x$.

g) State the domain for each of the graphs of the form $y = b^x$.

h) State the range for each of the graphs of the form $y = b^x$.

i) State the equation of the horizontal asymptote for each of the graphs of the form $y = b^x$. 
Warm-Up #5  Exploring the Value of $a$ in $y = ab^x$.

Consider the following functions with equations of the form $y = ab^x$:

i) $y = 3^x$
ii) $y = (2)3^x$
iii) $y = (5)3^x$
iv) $y = (0.5)3^x$

a) Graph the functions using a graphing calculator. Sketch the graphs on the grid.

b) Complete the following table for these equations of the form $y = ab^x$.

<table>
<thead>
<tr>
<th>Value of $a$</th>
<th>$y = 3^x$</th>
<th>$y = (2)3^x$</th>
<th>$y = (5)3^x$</th>
<th>$y = (0.5)3^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of $b$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y$-intercept of the graph</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c) What is the effect on the graph of changing the value of $a$?

d) If $f(x) = 3^x$, write the following in terms of the function $f$.

i) $y = 3^x$  ii) $y = (2)3^x$  iii) $y = (5)3^x$  iv) $y = (0.5)3^x$

e) Which transformations are associated with d): ii), iii), iv).

f) What is the effect of the parameter $a$ on the graph of $y = ab^x$?

g) State the $y$-intercept for each of the graphs of the form $y = ab^x$. 
Characteristics of the Graph of the Exponential Function $f(x) = ab^x$

The following summarizes the basic characteristics of the graph of the exponential function with equation $y = ab^x$.

Use the information from the previous Warm-Ups to complete the following.

- The y-intercept is _____ .  • There is _____ x-intercept.
- The x-axis is a ______________ ______________.
- The domain is ___________________________.
- The range is ____________________________.
- For $a > 0$,
  - when $b > 1$, the function represents a ___________ function.
  - when ____________, the function represents a decay function .
- The value of $b$ affects the steepness of the graph as $x$ increases.
  - when $b > 1$, the curve ___________ sharply as $b$ increases.
  - when $0 < b < 1$, the curve ____________ sharply as $b$ decreases.
- The value of $a$ affects the vertical stretch of the graph, namely:
  - when $a > 1$, the stretch is an ___________.
  - when $0 < a < 1$, the stretch is a ____________.
  - when $a < 0$, there is also a ________________________________.

Describe how the graph of the second function compares to the graph of the first function.

a) $y = 4^x$
   
   $y = 2(4)^{x-2}$

b) $y = 2^x$
   
   $y + 4 = -2^{\frac{x}{5}}$

Explain using transformations why the graph of $y = \left(\frac{1}{3}\right)^x$ is a reflection in the y-axis of the graph of $y = 3^x$.

Complete Assignment Questions #1 - #8
Assignment

1. State the x and y-intercepts for the graphs of the following:
   a) \( f(x) = 2^x \)
   b) \( f(x) = (2)10^x \)
   c) \( f(x) = 2^{10x} \)
   d) \( y = \left(-\frac{1}{2}\right)\left(\frac{3}{5}\right)^x \)

2. a) State the domain and range of the function \( f(x) = ab^x, a, b > 0, x \in \mathbb{R} \).

   b) Which of the following transformations applied to the graph of \( y = ab^x, a, b > 0, x \in \mathbb{R} \), would result in a change to the domain of the function?
      i) horizontal expansion
      ii) vertical compression
      iii) horizontal translation
      iv) reflection in the x-axis
      v) reflection in the y-axis
      vi) reflection in the line \( y = x \)
      vii) reciprocal transformation
      viii) absolute value transformation.

   c) Which of the above transformations applied to the graph of \( y = ab^x, a, b > 0, x \in \mathbb{R} \), would result in a change to the range of the function?

3. The graph of the exponential function with equation \( y = b^x \) is shown.

   a) Write down the equation of the function after \( y = b^x \) is reflected in the y-axis. Sketch this transformation on the grid.

   b) Write down the equation of the function after a reciprocal transformation of \( y = b^x \). Sketch this transformation on the grid.

   c) What do you notice? Explain algebraically.

   d) Would your observations in c) be the same if the original function had equation \( y = ab^x, a \neq 1 \)? Explain.
4. The graph of the exponential function with equation \( y = 4^x \) is shown.

a) Use the graph to estimate, to one decimal place, the solution to the equation \( 4^x = 12 \).

b) Use a graphing calculator to determine, to one decimal place, the solution to the equation \( 4^x = 12 \).

c) On the grid sketch the graph of the function with equation \( y = \left( \frac{1}{4} \right)^x \).

d) Without using the grid or a graphing calculator, state the solution to the equation \( \left( \frac{1}{4} \right)^x = 12 \).

e) Use transformations to sketch the graph of \( y = \left( \frac{1}{4} \right)^{x-2} - 4 \).

f) State the domain and range of the function \( f(x) = \left( \frac{1}{4} \right)^{x-2} - 4 \).

g) State the domain and range of the function \( f(x) = b^{x-h} + k \).

h) State the equation of the horizontal asymptote of the graph of the function \( f(x) = b^{x-h} + k \).
5. Describe how the graph of the second function compares to the graph of the first function.
   a) \( y = 10^x \)  
   b) \( y = 2^x \)  
   c) \( y = 6^x \)  
   d) \( y = a^x \)

   \[ y = 10^{-x} - 3 \]  
   \[ y = 5\left(\frac{1}{2}\right)^x \]  
   \[ y = \left(\frac{1}{6}\right)^{-x} \]  
   \[ y = a^\frac{x}{2} \]

6. The graph of \( f(x) = b^x \) is shown. Sketch \( f^{-1}(x) \).
   a) State the domain and range of \( f(x) \).
   b) State the domain and range of \( f^{-1}(x) \).
   c) State the asymptotes for \( y = f(x) \) and \( y = f^{-1}(x) \).
   d) Write the equation for the inverse function in the form \( x = f(y) \) and try to solve for \( y \). Explain what happens.

7. Which equation represents an exponential function?
   A. \( y = 2x^8 \)  
   B. \( y = (-3)^x \)  
   C. \( y = \frac{3^{x-2}}{2} \)  
   D. \( y = \frac{1}{3x} \)

8. Which equation determines an asymptote of \( y = 3^{x-2} + 1 \)?
   A. \( y = 3 \)  
   B. \( y = 2 \)  
   C. \( y = 1 \)  
   D. \( y = -1 \)

9. The graph of \( g(x) = \frac{1}{f(x)} \) is shown. If \( f(x) \) is an exponential function with equation \( y = ab^x \), \( a, b > 0 \)
   the value of \( a \) to one decimal place is: _____.
Answer Key

1. a) $x$-intercept: none   b) $x$-intercept: none   c) $x$-intercept: none   d) $x$-intercept: none
   $y$-intercept: 1   $y$-intercept: 2   $y$-intercept: 1   $y$-intercept: $-\frac{1}{2}$

2. a) Domain: $x \in \mathbb{R}$  Range: $\{y \mid y > 0, y \in \mathbb{R}\}$
   b) vi
   c) iv, vi

3. a) $y = b^{-x}$
   b) $y = \frac{1}{b^x}$
   c) identical since $b^{-x} = \frac{1}{b^x}$
   d) No since $ab^{-x} \neq \frac{1}{a b^x}$

4. a) 1.8   b) 1.8
   c)
   d) $-1.8$
   e)

5. a) reflection in the $y$-axis, and a vertical translation 3 units down.
   b) reflection in the $y$-axis, and a vertical expansion by a factor of 5
   c) identical
   d) horizontal expansion by a factor of 2 about the $y$-axis

6. a) Domain: $x \in \mathbb{R}$  Range: $\{y \mid y > 0, y \in \mathbb{R}\}$  b) Domain: $\{x \mid x > 0, x \in \mathbb{R}\}$  Range: $y \in \mathbb{R}$
   c) for $y = f(x)$ the asymptote is $y = 0$  for $y = f^{-1}(x)$ the asymptote is $x = 0$
   d) $x = b^{y}$  don’t know how to solve for $y$ until the next lesson

7. C   8. C   9. 0.2
Exponential and Logarithmic Functions Lesson #3: The Exponential Function
Exponential and Logarithmic Functions Lesson #4: The Logarithmic Function

Warm-Up #1

In Lesson 3, assignment question 6d), we were asked to find the inverse of \( y = b^x \) and solve for \( y \). In this lesson we will learn how to do this.

Logarithmic Function

A logarithmic function is the inverse of an exponential function. Remember, to find the inverse of a function we must switch \( x \) and \( y \) and then solve for \( y \). But for the inverse of an exponential function it is difficult to solve for \( y \). For example, to get the inverse of \( y = 2^x \) the first step is \( x = 2^y \), but it is difficult to solve for \( y \). To solve for \( y \) we introduce the logarithmic function as follows:

\[
x = 2^y \quad \Rightarrow \quad y = \log_2 x
\]

Therefore we write \( y = \log_b x \) rather than \( x = b^y \) to express the inverse of \( y = b^x \).

The logarithmic function with base \( b \) has the equation

\[
y = \log_b x, \quad x > 0, \ x \in \mathbb{R}, \ b > 0 \text{ and } b \neq 1.
\]

Warm-Up #2 Comparing the Graphs of \( y = 2^x \) and \( y = \log_2 x \)

a) Construct the graph of the exponential function \( y = 2^x \) and its inverse \( y = \log_2 x \), \( (x = 2^y) \), \( x, y \in \mathbb{R} \), using the grid and tables of values provided below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

b) Complete the table below

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>( x )-intercept</th>
<th>( y )-intercept</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = 2^x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \log_2 x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
c) Use the table of values to complete the following.

<table>
<thead>
<tr>
<th>Logarithmic Form</th>
<th>Exponential Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_2{8} = 3 )</td>
<td>( 8 = 2^3 )</td>
</tr>
<tr>
<td>( \log_2{4} = )</td>
<td></td>
</tr>
<tr>
<td>( \log_2{2} = )</td>
<td></td>
</tr>
<tr>
<td>( \log_2{1} = )</td>
<td></td>
</tr>
<tr>
<td>( \log_2{\frac{1}{2}} = )</td>
<td></td>
</tr>
<tr>
<td>( \log_2{\frac{1}{4}} = )</td>
<td></td>
</tr>
<tr>
<td>( \log_2{\frac{1}{8}} = )</td>
<td></td>
</tr>
</tbody>
</table>

- The logarithms are the exponents in the function \( y = 2^x \).
- Since the logarithmic function \( y = \log_b{x} \) is only defined for positive values of \( x \), the logarithm of a negative number cannot be determined.

**Characteristics of the Graph of the Logarithmic Function \( y = \log_b{x} \)**

- The \( x \) intercept is 1.
- There is no \( y \)-intercept.
- The \( y \)-axis is a vertical asymptote with equation \( x = 0 \).
- Domain = \( \{x \mid x > 0, x \in R\} \).
- Range = \( \{y \mid y \in R\} \).
- \( y = \log_b{x} \) is equivalent to \( x = b^y \), where \( x > 0 \) and \( b > 0, b \neq 1 \).
- \( b \) is the base of both the logarithmic function and the exponential function.
- The logarithmic equation \( y = \log_b{x} \) can be expressed in exponential form as \( x = b^y \).
- The exponential equation \( y = b^x \) can be expressed in logarithmic form as \( \log_b{y} = x \).
Class Ex. #1

Convert each of the following from logarithmic form to exponential form.

a) \( \log_7 x = 4 \)

b) \( \log_{10} 1000 = 3 \)

c) \( m = \log_B B \)

d) \( \log_b a = b^d \)

e) \( 5 = 4 \log_b 6 \)

Class Ex. #2

Convert each of the following from exponential form to logarithmic form.

a) \( 4^3 = 64 \)

b) \( 2^{-3} = \frac{1}{8} \)

c) \( e^d = f \)

d) \( x^{2y} = 5 \)

e) \( a = (2x + 4)^{-1} \)

Class Ex. #3

Solve for \( y \).

a) \( \log_3 81 = y \)

b) \( y = \log_5 \sqrt{125} \)

c) \( y = 2 \log_8 512 \)

Class Ex. #4

a) What is the value of \( \log_b 1 \)? Explain.

b) What is the value of \( \log_b b \)? Explain.

Class Ex. #5

Evaluate.

a) \( \log_4 64 \)

b) \( \log_2 \left( \frac{1}{32} \right) \)

c) \( 5^{\log_5 25} \)

d) \( a^{\log_a a} \)
Find the inverse of the following equations. Answer in the form $y = \underline{\hspace{2cm}}$.

a) $y = \log_3 x$

b) $y = 8^x$

---

**Complete Assignment Questions #1 - #9**

**Warm-Up #3**

We have seen how to change forms between the exponential form $y = b^x$ and the logarithmic form $\log_b y = x$.

We now consider how to write the exponential form $y = ab^x$ in logarithmic form.

This can be done using the following procedure:

1. Write the exponential form $y = ab^x$ as $\frac{y}{a} = b^x$.

2. Change $\frac{y}{a} = b^x$ to logarithmic form.

The logarithmic form of $y = ab^x \left( \text{or} \frac{y}{a} = b^x \right)$ is $\underline{\hspace{2cm}}$.

---

Change each of the following from exponential form to logarithmic form.

a) $y = 2(3^x)$

b) $h = 7(4)^k$

c) $5y = 30^x$

d) $y = \frac{3}{2}(10)^x$

e) $t = r(s)^p$

---

Change each of the following from logarithmic form to exponential form $y = ab^x$.

a) $\log_7 \left( \frac{y}{3} \right) = x$

b) $\log_{10} \left( \frac{y}{4} \right) = x$

c) $\log_5 (7y) = x$

d) $\log_e \left( \frac{y}{5} \right) = x$
Assignment

1. Complete the following from the graphs of \( y = b^x \) and \( y = \log_b x \), \( b > 0 \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>x-intercept</th>
<th>y-intercept</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = b^x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \log_b x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Why does \( x \) have to be greater than zero in the domain of \( y = \log_b x \), and not in \( y = b^x \), \( b > 0 \)?

3. Express each of the following in logarithmic form.
   a) \( 5^2 = 25 \)  
   b) \( 3^0 = 1 \)  
   c) \( 2^{-4} = \frac{1}{16} \)  
   d) \( \left( \frac{1}{2} \right)^4 = \frac{1}{16} \)  
   e) \( b^d = e \)

4. Express each of the following in exponential form.
   a) \( \log_3 9 = 2 \)  
   b) \( \log_5 625 = 4 \)  
   c) \( \log_4 \frac{1}{4} = -1 \)  
   d) \( \log_a f = i \)  
   e) \( \log_{10} 0.001 = -3 \)
5. Evaluate.
   a) \( \log_{4} 64 \)  
   b) \( \log_{5} \sqrt{5} \)  
   c) \( \log_{7} 49 \)  
   d) \( \log_{10} 0.001 \)  
   e) \( \log_{8} 8^{-4} \)  
   f) \( \log_{2} \sqrt{\frac{1}{512}} \)  
   g) \( \log_{b} 1 \)  
   h) \( \log_{c} c \)  
   i) \( \log_{x} x^{2} \)  

6. Complete the following table:

<table>
<thead>
<tr>
<th>Logarithmic Form</th>
<th>Exponential Form</th>
<th>Value of ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_{4} x = 2 )</td>
<td>7 = 49^x</td>
<td></td>
</tr>
<tr>
<td>( \log_{5} \left( \frac{1}{64} \right) = -3 )</td>
<td>( x + 2 = 4^2 )</td>
<td></td>
</tr>
<tr>
<td>( \log_{3} 2x = \frac{1}{5} )</td>
<td>( \frac{1}{2} = 16^x )</td>
<td></td>
</tr>
</tbody>
</table>

7. Is \( y = \log_{3} x \) the logarithmic form of \( y = 3^x \)? Explain your answer.
8. Solve for $x$.
   a) $\log_{x}125 = 3$
   b) $\log_{125}5 = x$
   c) $\log_{4}x = -8$

9. Find the inverse of the following equations. Answer in the form $y = \ldots$.
   a) $y = 3^x$
   b) $y = \log_{4}x$
   c) $y = 3x^2 + 2$
   d) $y = \log_{3}x$
   e) $y = 20^x$
   f) $x = 20^y$

10. Change each of the following from exponential form to logarithmic form.
    a) $y = 3(2)^x$
    b) $y = 10(3)^x$
    c) $7y = 30^x$
    d) $y = \frac{5}{6}(10)^x$
    e) $a = b(c)^d$
    f) $2y = 7\left(\frac{3}{4}\right)^x$

11. Change each of the following from logarithmic form to exponential form $y = ab^x$.
    a) $\log_{8}\left(\frac{y}{9}\right) = x$
    b) $\log_{20}(6y) = x$
    c) $\log_{e}\left(\frac{y}{5}\right) = x$
    d) $\log_{10}(0.5y) = x$
12. Solve the following equations for \( y \).

\[
a) \quad 3 = \log_2 \left( \frac{y}{4} \right) \quad \text{b) } \quad \log_2 \left( \frac{y}{5} \right) = -3 \quad \text{c) } \quad 2 = \log_4 32y
\]

**Multiple Choice**

13. If \( \log_4 (4096x) = 64 \), then the value of \( x \) is

A. \( 4^{\frac{32}{3}} \)  
B. \( 4^{58} \)  
C. \( 4^6 \)  
D. \( 4^{32} \)

14. If \( \log_2 x = 3 \) and \( \log_2 t = x \), then \( t \) equals

A. 64  
B. 128  
C. 256  
D. 512

**Numerical Response**

15. If \( \log_b 81 = \frac{2}{3} \), then the value of \( b \) is to the nearest whole number is: ______.

**Numerical Response**

16. If \( \log_a b = 4.5 \) and \( \log_a c = 3.7 \), then the value of \( \log_a \left( \frac{b}{c} \right) \) to the nearest tenth is: ______.
### Answer Key

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>$x$-intercept</th>
<th>$y$-intercept</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = b^x$</td>
<td>$x \in \mathbb{R}$</td>
<td>${y \mid y &gt; 0, \ y \in \mathbb{R}}$</td>
<td>none</td>
<td>1</td>
<td>$y = 0$</td>
</tr>
<tr>
<td>$y = \log_b x$</td>
<td>${x \mid x &gt; 0, \ x \in \mathbb{R}}$</td>
<td>$y \in \mathbb{R}$</td>
<td>1</td>
<td>none</td>
<td>$x = 0$</td>
</tr>
</tbody>
</table>

2. $y = \log_b x \Rightarrow x = b^y$

$b^y$ must be greater than zero, so $x$ must be greater than zero.

$y = b^x$ can be determined for all values of $x$ positive, negative, or zero.

3. a) $\log_5 25 = 2$  
   b) $\log_3 1 = 0$  
   c) $\log_2 \left(\frac{1}{16}\right) = -4$  
   d) $\log_4 \left(\frac{1}{16}\right) = 4$  
   e) $\log_b e = d$

4. a) $3^2 = 9$  
   b) $5^4 = 625$  
   c) $4^{-1} = \frac{1}{4}$  
   d) $a^f = f$  
   e) $10^{-3} = 0.001$

5. a) 3  
   b) $\frac{1}{2}$  
   c) 2  
   d) -3  
   e) -4  
   f) $-\frac{9}{2}$  
   g) 0  
   h) 1  
   i) $z$

6. | Logarithmic Form | Exponential Form | Value of $x$ |
|-----------------|-----------------|-------------|
| $\log_4 x = 2$  
| $\log_{49} 7 = x$  
| $\log_3 \left(\frac{1}{64}\right) = -3$  
| $\log_4 (x + 2) = 2$  
| $\log_{32} x = \frac{1}{5}$  
| $\log_{10} \left(\frac{1}{2}\right) = x$ | $x = 4^2$ | $x = \frac{1}{2}$ |
| $7 = 49^x$  
| $\frac{1}{64} = x^{-3}$  
| $x + 2 = 4^2$  
| $x = 32^{\frac{1}{5}}$  
| $\frac{1}{2} = 16^x$ | $x = 4$ | $x = 2$ |

7. No, it is the inverse. The log form of $y = 3^x$ is $x = \log_3 y$.

8. a) 5  
   b) $\frac{1}{3}$  
   c) 0.0000153

9. a) $y = \log_3 x$  
   b) $y = 4^x$  
   c) $y = \pm \sqrt[3]{\frac{x - 2}{3}}$  
   d) $y = 3^x$  
   e) $y = \log_{20} x$  
   f) $y = 20^x$

10. a) $\log_2 \left(\frac{y}{3}\right) = x$  
    b) $\log_3 \left(\frac{y}{10}\right) = x$  
    c) $\log_{30} (7y) = x$  
    d) $\log_{10} \left(\frac{6y}{5}\right) = x$  
    e) $\log_e \left(\frac{a}{b}\right) = d$  
    f) $\log_{\frac{2}{7}} \left(\frac{2y}{7}\right) = x$

11. a) $y = 9(8)^x$  
    b) $y = \frac{1}{6}(20)^x$  
    c) $y = 5(e)^x$  
    d) $y = 2(10)^x$

12. a) 32  
    b) $\frac{5}{8}$  
    c) $\frac{1}{2}$  
    13. B  
    14. C  
    15. 729  
    16. 0.8
Exponential & Logarithmic Functions Lesson #5: Changing the Base of Logarithms

**Warm-Up #1**

In the previous lesson we discussed logarithms with many different bases. In order to do numerical calculations on a calculator, we use two specific bases - base 10 and base \( e \).

### Common Logarithms

*Common logarithms* are logarithms in base 10, eg. \( \log_{10}1000 \). These logarithms are in such common use that when a base is not given the logarithm is understood to be in base ten. For instance,

\[ \log_{10}1000 \text{ is often written as } \log 1000 \]

On a graphing calculator, this can be evaluated using the LOG key.

**Class Ex. #1**

Evaluate each of the following logarithms manually and by calculator.

a) \( \log 1000 \)  

b) \( \log \sqrt[3]{1000} \)

### Natural Logarithms

*Natural logarithms* are logarithms in base \( e \) eg. \( \log_e 15 \). \( e \) is an irrational number, the value of which will be determined in Warm-Up #2.

\[ \log_e 15 \text{ is often written as } \ln 15 \]

On a graphing calculator, this can be evaluated using the LN key.

**Class Ex. #2**

Evaluate the following logarithms to one decimal place where necessary.

a) \( \ln 5 \)  

b) \( \log_e 5 \)  

c) \( \ln e \)
Warm-Up #2 is not required for this course, but may give students a greater understanding in preparation for higher level math courses.

**Warm-Up #2**

**Approximating the Value of e**

The formal definition of the irrational number $e$ is the limit as $x$ approaches infinity of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$. Complete the following table to determine the value of this function as $x$ gets very large. Use the TABLE feature of a graphing calculator and work to 4 decimal places.

<table>
<thead>
<tr>
<th>$x$</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10 000</th>
<th>100 000</th>
<th>1 000 000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Estimate for $e$ is __________________.

A more accurate estimate can be determined by pressing the $e$ key on a graphing calculator.

The value of $e$ to 9 decimal places is ___________________________.

**Warm-Up #3**

a) Evaluate $\log_5 25$

b) Try to evaluate $\log_5 50$. What problem do you encounter?

At the moment we are unable to evaluate $\log_5 50$, but by converting to common logarithms, or natural logarithms, we can use a calculator to determine the value of $\log_5 50$. The method for converting from one base to another is discussed in Warm-Up #4.
Warm-Up #4  

**Change of Base**

a) Evaluate.

i) \( \log_5 25 \)  

ii) \( \frac{\log 25}{\log 5} \)  

iii) \( \frac{\log_e 25}{\log_e 5} \)

b) Evaluate.

i) \( \log_3 243 \)  

ii) \( \frac{\log 243}{\log 3} \)  

iii) \( \frac{\log_e 243}{\log_e 3} \)

c) Write \( \log_2 64 \) in a form which can be evaluated using a calculator.

**Change of Base Identity**

\[ \log_b c = \frac{\log_a c}{\log_a b} \]

*This formula is NOT on the formula sheet*

We have seen in Warm-Up #4 that the above identity is true for converting logarithms to base 10 or base \( e \). In fact it holds true for converting logarithms to any base. The example below supports this.

i) Evaluate \( \log_4 1024 \)  

ii) Evaluate \( \frac{\log_2 1024}{\log_2 4} \)

We will be able to prove this identity in a later lesson when we have developed an understanding of the laws of logarithms.

---

Class Ex. #3

Evaluate the following logarithms to the nearest hundredth by changing the base.

a) \( \log_5 221 \)  

b) \( \log_2 \frac{1}{1000} \)  

c) \( 3 \log_7 512 \)
Class Ex. #4

Convert the following logarithms to the base indicated.

a) \( \log_{6}216 \) to base 3  
   b) \( \log 300 \) to base 5

Class Ex. #5

Find the exact value of the following.

a) \( \log_{3} \frac{1}{729} \)  
   b) 2\( \log_{8}512 \)  
   c) \( -\log_{7} \left( \frac{1}{343} \right) \)

   d) \( 5^{\log_{5}25} \)  
   e) \( \log_{2} \sqrt{\frac{1}{1024}} \)  
   f) \( \log_{7}49^{-5} \)

Complete Assignment Questions #1 - #10

Assignment

1. Evaluate each of the following logarithms.
   a) \( \log 100 \)  
   b) \( \log 10^6 \)  
   c) \( \log \sqrt{10} \)  
   d) \( \log 0.01 \)

2. Evaluate the following logarithms to the nearest tenth.
   a) \( \ln 20 \)  
   b) \( \log_{e}8 \)  
   c) \( \ln e^2 \)

3. Convert the following logarithms to the base indicated.
   a) \( \log_{8}35 \) to base 7  
   b) \( \log_{\frac{1}{2}} \) to base 6  
   c) \( \log_{5}50 \) to base e
4. Evaluate using the change of base identity to the nearest hundredth:
   a) \( \log_{5} 17 \)  
   b) \( \log_{0.5} 5.9 \)  
   c) \( \frac{1}{\log_{5} 3} \)  
   d) \(-2\log_{12} 6\)  
   e) \( \log_{8} 8 \)

5. Evaluate each expression:
   a) \( 4^{\log_{4} 4} \)  
   b) \( 10^{\log_{10} 1000} \)

6. Describe how to graph \( y = \log_{3} x \) using a graphing calculator. Sketch the graph and determine the \( x \)-intercept.

7. Which of the following has a negative value?
   
   A. \( -\log_{4}(0.1) \)  
   B. \( \log_{4} \left( \frac{5}{2} \right) \)  
   C. \( \log_{4} \left( \frac{2}{3} \right) \)  
   D. \( \log_{4} \left( \frac{2}{3} \right) \)

8. The value of the expression \( \log_{\sqrt{2}} 8 + 2\log_{9} 3 \) to the nearest tenth is _____.

9. Given the equation \( \log_{7} x = \log_{4} 60 \), the value of \( x \) to the nearest whole number is _____.

10. If \( \log_{x} 27 = \log_{12} 3 \), the value of \( x \) to the nearest whole number is _____.
Answer Key

1. a) 2 b) 6 c) $\frac{1}{2}$ d) $-2$

2. a) 3.0 b) 2.1 c) 2.0

3. a) $\frac{\log 35}{\log 8}$ b) $\frac{\log (\frac{1}{2})}{\log 10}$ c) $\frac{\log 50}{\log 3}$

4. a) 1.76 b) $-2.56$ c) 1.46 d) $-1.44$ e) 1.00

5. a) 4 b) 1000

6. Graph $y = \frac{\log x}{\log 5}$, x-intercept is 1

7. D

8. 7.0

9. 313

10. 1728
Exponential and Logarithmic Functions Lesson #6: Laws of Logarithms

Warm-Up #1
Evaluate the following.

i) \( \log_2 16 + \log_2 8 \)

ii) \( \log_2 [(16)(8)] \)

What do you notice?

Warm-Up #2
Evaluate the following log equations

i) \( \log_2 16 - \log_2 8 \)

ii) \( \log_2 \frac{16}{8} \)

What do you notice?

Product and Quotient Laws of Logarithms
Warm-Ups #1 and #2 are examples of the following laws

\[
\begin{align*}
\log_a (M \times N) &= \log_a M + \log_a N & \text{The Product Law} \\
\log_a \left( \frac{M}{N} \right) &= \log_a M - \log_a N & \text{The Quotient Law}
\end{align*}
\]

These formulas are NOT on the formula sheet

The laws of logarithms are identities which will be proved in Lesson 9.

Class Ex. #1
Evaluate the following using the product law or quotient law.

a) \( \log_2 12 - \log_2 3 \)

b) \( \log_6 9 + \log_6 8 - \log_6 2 \)

c) \( \log_5 10 + \log_5 75 - (\log_5 2 + \log_5 3) \)

d) \( \log_2 2 + \log_2 3 - \log_2 6 - \log_2 8 \)
The expression \( \log_2 x + \log_2 2x - \log_2 x^2 - \log_2 y \) is equivalent to
A. \( 2 + \log_2 y \)
B. \( 1 + \log_2 y \)
C. \( 2 - \log_2 y \)
D. \( 1 - \log_2 y \)

Evaluate.

a) \( \log_2 15 + \log_2 14 - \log_2 105 \)  
b) \( \log_4 2^8 + \log_4 \left( \frac{1}{8} \right)^2 \)

Complete Assignment Questions #1 - #3

Warm-up #3

Show that \( 2\log_3 x = \log_3 x^2 \).

The Power Law of Logarithms

Warm-Up #3 is an example of the power law of logarithms.

\[
\log_a M^n = n \log_a M \quad \text{The Power Law}
\]

This formula is NOT on the formula sheet

Use the laws of logarithms to evaluate each of the following.

a) \( \frac{1}{2} \log_2 16 - \frac{1}{3} \log_2 8 \)  
b) \( 2 \log 5 + 2 \log 2 \)

c) \( 3\log x - \log x^3 \)  
d) \( \log_3 \frac{27^3}{81^4} \)
Simplify the following.

a) \( \log_6 6^n \)  

b) \( 6^{\log_6 n} \)

Class Ex. #5 is an example of the following logarithmic identities:

- \( \log_b b^n = n \) and \( b^{\log_b n} = n \)

These identities follow from the fact that the logarithmic and exponential functions are inverses.

Write the following expression as a single logarithm.

\( \log B + \log D - 5\log E - \log A^2 + \frac{1}{2}\log A \)

Write the following expression as a single logarithm.

\( \frac{3}{2} (\log_b x + 2\log_b y^4) - \frac{1}{2} \left( \log_b \sqrt{x} + \log_b y^\frac{4}{3} \right) \)

Complete Assignment Questions #4 - #14
Assignment

1. Use the laws of logarithms to evaluate each of the following.
   a) \( \log_2 4 + \log_2 8 \)
   b) \( \log_4 \left( \frac{7}{2} \right) - \log_4 56 \)
   c) \( \log_6 9 + \log_6 8 - \log_6 2 \)
   d) \( \log 2 + \log 10 - \log \frac{1}{5} \)

2. Use the laws of logarithms to evaluate each of the following.
   a) \( \log_2 \left( \frac{4}{3} \right) + \log_2 768 \)
   b) \( \log 8 + \log 5 - \log \frac{2}{5} \)
   c) \( \log 3 + \log 4 + \log \frac{1}{2} + \log \frac{1}{6} \)
   d) \( \log_3 3 + \log_3 2 - \log_3 27 - \log_3 6 \)
   e) \( \log_3 9 - \log_3 \left( \frac{1}{3} \right) \)
   f) \( \log_2 (2^8) + \log_2 \left( \frac{1}{8} \right)^2 \)
   g) \( \log_a 1 + \log_a 1 + \log_a 1 \)
3. Which of the following are true and which are false for logarithms to 
ev
ey every base?

a) \( \log 2 + \log 3 = \log 5 \quad b) \ \log 3 + \log 4 = \log 12 \quad c) \ \log 8 = \log 4 + \log 2 \)

d) \( \log 10 + \log 10 = \log 100 \quad e) \ \log 2 \times \log 3 = \log 6 \quad f) \ \frac{\log 8}{\log 2} = \log 4 \)

g) \( \log 3^2 + \log 3^{-2} = 0 \quad h) \ \log \frac{5}{3} = \frac{\log 5}{\log 3} \quad i) \ \log \frac{1}{8} = -\log 8 \)

4. Use the laws of logarithms to evaluate the following.

a) \( \log_2 \sqrt{16} \quad b) \ \log_3 27^{\frac{1}{2}} \)

5. Write each expression as a single logarithm:

a) \( \log x - 3 \log y - 2 \log z \quad b) \ \frac{1}{3} \log_a p + 3 \log_a q - 4 \log_a p \)

6. Simplify the following:

a) \( \log 2 + 2 \log 3 - \log 18 \quad b) \ 2\log_4 2 - 2\log_4 4 - \log_4 \frac{1}{4} \)

7. a) Show that \( \log 81 = 4 \log 3 \).

b) Hence simplify: \( (i) \ \log 81 - \log 27 \quad (ii) \ \frac{\log 81}{\log 27} \)
8. Which of the following are true and which are false for logarithms to every base?

a) $\log 5^{-2} = -2 \log 5$  

b) $\log 4 = \frac{2}{3} \log 8$  

c) $\log 27 = \frac{3}{4} \log 81$

d) $\frac{1}{3} \log 11 = \log \frac{11}{3}$  

e) $\log 5 = \frac{1}{2} \log 10$  

f) $\frac{\log \sqrt{2}}{\log \sqrt{8}} = \frac{1}{3}$

g) $\log \frac{1}{5} - \log 5 = - \log 25$

h) $\log 2 - \log \sqrt{2} = \log \sqrt{2}$

9. Which is the greatest of $\frac{2}{3} \log 1$, $\frac{3}{4} \log 1$, $\frac{4}{3} \log 1$?

10. Simplify the following:

a) $\log x^4 - 3 \log x + \log \frac{1}{x}$  

b) $\log x^{\frac{1}{2}} + \log y^{\frac{1}{2}} = \frac{1}{2} \log xy$

c) $\log_2 \sqrt{6} - \frac{1}{2} \log_2 3$  

d) $\frac{1}{2} \log_{10} 10 + 3 \log_{10} \sqrt{10}$

e) $\log_e a + \log_e a^5 - \log_e a$

f) $\frac{2}{3} \log_2 a - 5 \log_2 b - \frac{1}{3} \log_2 c^3$
g) \( \log_{e}y^{2x-3} + \log_{e}y^{5x-2} - \log_{e}y^{x-5} - 2\log_{e}y^{3x} + 1 \)

11. Simplify.

a) \( \log_{5}5^{7} \) 

b) \( 10^{\log_{6}} \) 

c) \( \ln e^{4} \) 

d) \( \log_{e}c^{t} \)

e) \( e^{\ln 7} \) 

f) \( (5^{\log_{5}2})(5^{\log_{5}3}) \) 

g) \( \frac{(\sqrt{2} \log_{6}27)(\sqrt{2} \log_{6}16)}{\sqrt{2} \log_{6}12} \)

12. The expression \( 3\log_{x}4 + \log_{x}8 - \frac{1}{4} \log_{x}16 \), where \( x > 0 \), is equal to

A. \( \log_{x}384 \)

B. \( \frac{3}{4} \log_{x}512 \)

C. \( \log_{x}256 \)

D. \( \frac{1}{4} \log_{x}\left(\frac{1}{2}\right) \)
13. If $X > 0$, $Y > 0$, and $Z = X^3 Y$, then $\log Y$ is equal to

A. $3 \log X - \log Z$
B. $\log Z - 3 \log X$
C. $\log 3X - \log Z$
D. $\log Z - \log 3X$

Answer Key
1. a) 5   b) −2   c) 2   d) 2
2. a) 10   b) 2   c) 0   d) −3   e) 3   f) 2   g) 0
3. a) F   b) T   c) T   d) T   e) F   f) F   g) T   h) F   i) T
4. a) $\frac{4}{3}$   b) $\frac{3}{2}$
5. a) $\log \left( \frac{x}{(y^2 z^2)^{\frac{3}{2}}} \right)$   b) $\log_a \frac{q^3}{p^\frac{11}{5}}$
6. a) 0   b) 0
7. b) (i) $\log 3$   (ii) $\frac{4}{3}$
8. a) T   b) T   c) T   d) F   e) F   f) T   g) T   h) T
9. none because each of these equal zero.
10. a) 0   b) 0   c) $\frac{1}{2}$   d) 2   e) $1 + \log_a a^4$   f) $\log_2 \left( \frac{a^\frac{2}{3}}{b^\frac{1}{3} c^\frac{1}{3}} \right)$
    g) $-2 \log_a b$ or $\log_a \frac{1}{y^2}$
11. a) 7   b) 6   c) 4   d) $t$   e) 7   f) 6   g) 2
Exponential & Logarithmic Functions Lesson #7: 
Solving Exponential and Logarithmic Equations - Part 1

Warm-Up #1

In lesson 2, we solved exponential equations by using a common base. In some cases this is difficult to do and a different process is required to solve the equation.

Solving Exponential Equations without using a Common Base

This method involves taking the common logarithm of each side and using the laws of logarithms to solve the equation.

Class Ex. #1

Solve the following equations. Express $x$

• as an exact value and,
• to three decimal places.

a) $4^x = 12$

b) $5^{3x} = 3^{2x - 1}$

c) $2(3)^x - 2 = 7^x$

Complete Assignment Questions #1 - #2
Solving Logarithmic Equations using the Laws of Logarithms

There are many techniques for solving logarithmic equations. These will be developed in the following class examples.

A logarithmic expression is defined only for positive values of the argument. When we solve a logarithmic equation it is essential to verify that the solution(s) does not result in the logarithm of a negative number. Solutions that would result in the logarithm of a negative number are called extraneous, and are not valid solutions.

Class Ex. #2

Solve for $x$ in the following equations. Verify your solutions.

a) $\log x - \log 15 = \log 0.2$

b) $\log_5 (x + 1) + \log_5 (x - 3) = 1$

Class Ex. #3

Solve and verify.

a) $\log x^2 - \log x^3 = 10$

b) $(\log x)^2 - \log x^3 = 10$
Solve $2\log_3 x - \log_3(x + 3) - 3 = 0$ to two decimal places.

Solve $\log_4(x + 1) - \log_4(2x - 3) = \log_4 8$.

**Complete Assignment Questions #3 - #8**

**Assignment**

1. Find the value of $x$ to the nearest hundredth.
   
   a) $4^x = 60$
   b) $7^{x + 2} = 41$
   c) $4^{x + 1} = 5^x - 2$
   
   d) $5^{2 - x} = 2$
   e) $3(2^x) = 6^{x - 2}$
2. Find the exact value of \( x \).
   a) \( 3^x = 90 \)
   b) \( 5^{x - 3} = 40 \)
   c) \( 0.5^{x + 2} = 6^{x - 1} \)
   d) \( 2^{2x} = 6^{x - 3} \)
   e) \( 4(3)^{x - 4} = 7^x \)

3. Solve for the variable in each equation. Verify.
   a) \( \log_3 x - \log_3 3 = \log_3 30 \)
   b) \( \log_3 3y - \log_3 4 = \log_3 6 \)
   c) \( 2 \log y = \log 25 \)
4. Solve for the variable in each equation.

a) \[ \log_5 x - \log_5 (x - 1) = \log_5 3 \]

b) \[ \log_4 (x - 5) + \log_4 (x - 2) = 1 \]

c) \[ \log_8 (-x) + \log_8 (3 - x) = \log_8 10 \]

d) \[ \log_3 (3x - 1) - \log_3 (x - 1) = 4 \]
e) \( \log_5(7x - 1) - \log_5 x = \log_5 4 \)

f) \( \log(2x + 3) + \log(x + 2) - 1 = 0 \)

g) \( \log_{49}(m + 4) + \log_{49}(m - 2) = \frac{1}{2} \)

h) \( \log_2 x = 2 + \frac{1}{2} \log_2(x - 3) \)

5. Solve and verify.
   a) \( 2\log_3 n - \log_3 2n = 2 \)
   b) \( \log x + (\log x)^2 = 0 \)
c) \((\log x)^2 + \log x^{-1} - 12 = 0\)  

\(d) \ \log_4 x^2 = (\log_4 x)^2\)

\(e) \ (\log x)^2 - \log x^5 = 14\)

\(f) \ 2(\log_3 n)^3 - (\log_3 n)^2 = 0\)
g) \( 5^{\log_3 x} - \log_3 2 = 125 \)

h) \( 3(\log_3 x)^2 - 36 = \log_3 x^{23} \)

i) \( \frac{1}{2} \log_4 (y + 4) + \frac{1}{2} \log_4 (y - 4) = \log_4 3 \)

j) \( \log_2 (1 - w) - \log_2 (3 - w) = -1 \)
6. If $21 = 6^{3x}$, then the value of $x$ is

A. $\frac{\log 21}{\sqrt{3}}$

B. $\frac{\log 21}{3}$

C. $\frac{\log 3}{3}$

D. $\frac{\log 21}{\log \sqrt{6}}$

7. If $\log_4(2x + 1) + \log_4(x - 1) = \frac{1}{2}$, then the value(s) of $x$ is/are

A. $\frac{3}{2}, -1$

B. $\frac{3}{2}$ only

C. 3 only

D. $\frac{1}{2}, 3$

8. If $2\log x - \log 25 = \log 9 + \log x, x > 0$, then the value of $x$ to the nearest integer is _____.

Numerical Response
**Answer Key**

1. a) 2.95  
   b) −0.09  
   c) 20.64  
   d) 1.57  
   e) 4.26

2. a) \( \frac{\log 90}{\log 3} \)  
   b) \( \frac{\log 40 + 3 \log 5}{\log 5} \)  
   c) \( \frac{\log 5000}{\log 5} \)  
   d) \( \frac{2 \log 0.5 + \log 6}{\log 6 - \log 0.5} \)  
   e) \( \frac{\log 5}{\log 12} \)

3. a) 90  
   b) 8  
   c) 5  
   d) 3  
   e) \( \frac{1}{3} \)

4. a) \( \frac{3}{2} \)  
   b) 6  
   c) −2  
   d) \( \frac{40}{39} \)  
   e) \( \frac{1}{3} \)  
   f) \( \frac{1}{2} \)  
   g) 3  
   h) 4, 12

5. a) 18  
   b) \( \frac{1}{10} \)  
   c) \( \frac{1}{1000}, \frac{1}{1000} \)  
   d) 1, 16  
   e) \( \frac{1}{100}, 1000000 \)  
   f) \( 1, \sqrt{3} \)  
   g) 54  
   h) \( 3^3 \) or 0.2311, 39 or 19683  
   i) 5  
   j) −1

6. A  
7. B  
8. 225
Exponential & Logarithmic Functions Lesson #8: Solving Exponential and Logarithmic Equations - Part 2

**Solving Logarithmic Equations using a Change of Base**

Class Ex. #1

Solve \( \log_2 x + \log_4 x = 7 \) to the nearest tenth.

Class Ex. #2

If \( \log_4(1024) = x + y \) and \( \log_864 = x - y \), then what is the value of \( x \) and \( y \)?

Complete Assignment Questions #1 - #3
Solving Exponential/Logarithmic Equations with a Graphing Calculator

In previous math courses we learned two ways to solve equations using a graphing calculator.

i) Insert the left side of the equation into Y₁. Insert the right side of the equation into Y₂. Graph Y₁ and Y₂ and determine the x-coordinate of the point(s) of intersection.

ii) Rearrange the original equation so that all terms are on the left hand side and zero is on the right side. Insert the rearranged left side into Y₁, graph and determine the x-intercept(s) of the graph.

Consider the equation $4^x = 7$.

a) Describe how you would use a graphing calculator to find the solution to the equation $4^x = 7$ by finding a point of intersection.

b) Complete the following statement

The grid provided shows the window $x: [ , , ]$ $y: [ , , ]$

c) Use the given window to determine the value of $x$ to the nearest tenth.

d) Show the method on the first grid.

e) Describe how you would use a graphing calculator to find the solution to the equation $4^x = 7$ by finding the x-intercept.

f) Determine the value of $x$ to the nearest tenth.

g) Show the method on the second grid.

h) Solve the equation algebraically.
a) Explain how to use a graphing method to determine the value of $x$, to the nearest tenth, if $\log_3 x = 1.5$. State an appropriate window.

b) Determine the solution graphically and verify algebraically.

---

**Assignment**

1. Solve to the nearest tenth.
   
a) $\log_5 x + \log_{10} x = 4$
   
   b) $\log_3 x - \log_6 x = 2$
2. Determine the value of \( x \) and \( y \) if \( \log_3(243) = 2x + y \) and \( \log_264 = x - y \).

3. If \( \log_9{x^2} - \log_{10}y = 5 \) and \( \log_9{x^3} + 2\log_{10}y = 11 \), find \( x \) and \( y \).

4. Describe two graphing methods to determine the value of \( x \) if \( 5^x = 30 \). State an appropriate window. Determine the solution graphically to the nearest hundredth and verify algebraically.
5. Describe two graphing methods to determine the value of \( x \) if \( \log_8 x = 0.6 \). State an appropriate window. Determine the solution graphically and verify algebraically.

6. The solution to a problem is the intersection point in the first quadrant of the function with equations \( y = \log_4 x \) and \( y = 2^x - 6 \). Using a graphing calculator, the y-coordinate of this point, correct to the nearest tenth, is

A. 6.4  
B. 2.8  
C. 0.7  
D. 1.3

Multiple Choice

Answer Key

1. a) 44.2  b) 292.9  
2. \( x = \frac{11}{3} \) and \( y = -\frac{7}{3} \)  
3. \( x = 729 \) and \( y = 10 \)

4. Method 1:  
Use window \( x:[-1, 4, 1] \) \( y:[-5, 40, 5] \) ← may vary →  
Graph \( Y_1 = 5^x \)  
Graph \( Y_2 = 30 \)  
Determine the \( x \) value of the point of intersection using the TRACE and INTERSECT features of the calculator.  
Solution → 2.11

Method 2:  
Use window \( x:[-1, 5, 1] \) \( y:[-40, 20, 5] \)  
Graph \( Y_1 = 5^x - 30 \)  
Determine the \( x \)-intercept using the TRACE and ZERO features of the calculator.  
Solution → 2.11

5. Method 1:  
Use window \( x:[-1, 10, 1] \) \( y:[-1, 1, 0.5] \) ← may vary →  
Graph \( Y_1 = \frac{\log x}{\log 8} \)  
Graph \( Y_2 = 0.6 \)  
Determine the \( x \) value of the point of intersection using the TRACE and INTERSECT features of the calculator.  
Solution → 3.48

Method 2:  
Use window \( x:[-1, 10, 1] \) \( y:[-1, 1, 0.5] \)  
Graph \( Y_1 = \frac{\log x}{\log 8} - 0.6 \)  
Determine the \( x \)-intercept using the TRACE and ZERO features of the calculator.  
Solution → 3.48

6. C
Exponential and Logarithmic Functions Lesson #9: Identities

**Warm-Up**

In mathematics it is important to understand the difference between an equation and an identity.

- \(2x^2 + 3 = 11\) is an equation. It is only true for certain values of the variable \(x\). The solutions to this equation are -2 and 2.

- \((x + 1)^2 = x^2 + 2x + 1\) is true for all values of the variable \(x\) and is called an identity.

We have already met the following identities:

- **Change of Base formula** \[ \log_b c = \frac{\log_a c}{\log_a b} \]
- **Product Law** \[ \log_a (MN) = \log_a M + \log_a N \]
- **Quotient Law** \[ \log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N \]
- **Power Law** \[ \log_a M^n = n \log_a M \]

**Note**
- To prove an identity we must not assume that it is true and treat it like an equation. We must treat each side separately and show that both sides are identical.
- In all the identities in this unit, we assume that the bases are positive real numbers, not equal to one, and that the variables are in the domain of the logarithmic function.

**Class Ex. #1**

Use the following steps to prove the identity \(\log_b c = \frac{\log_a c}{\log_a b}\).

Let \(\log_b c = x\) \[ \log_b c = x \]

Change to exponential form

Take the logarithm, in base \(a\) of both sides

Use the power law

Solve for \(x\)

Replace \(x\) with \(\log_b c\)

We have shown that the left side = right side and so the identity is proved.
Use the following steps to prove the identity \( \log_a (M \times N) = \log_a M + \log_a N \).

Let \( \log_a M = x \) and \( \log_a N = y \)  

Change to exponential form

Form the product \( MN \)

Use the product law for exponents

Take the logarithm, in base \( a \) of both sides

Use the power law and simplify

Substitute for \( x \) and \( y \)

We have shown that the left side = right side and so the identity is proved.

**Complete Assignment Questions #1 - #2**

**Using a T-Table to Prove an Identity**

This technique ensures that we treat the left hand side and the right hand side separately.

Prove the identity \( \log x^3 + \log \left( \frac{1}{x} \right) = 2\log x \) and state the values of \( x \) for which the identity is defined.

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log x^3 + \log \left( \frac{1}{x} \right) )</td>
<td>( 2\log x )</td>
</tr>
</tbody>
</table>
Prove the identity \[ \frac{1}{\log_5 x} + \frac{1}{\log_2 x} = \frac{1}{\log_{10} x} \]

<table>
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<tr>
<th>Left Side</th>
<th>Right Side</th>
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Complete Assignment Questions #3 - #11

**Assignment**

1. Use the method in Class Ex. #2 to prove the identity \( \log_a \left( \frac{M}{N} \right) = \log_a M - \log_a N \).
2. Use the method in Class Ex. #2 to prove the identity $\log_a M^n = n \log_a M$.

3. Prove the identity $\log_a \left( \frac{1}{x} \right) = -\log_a x$

4. Prove the identity $(\log_a b)(\log_b c) = \log_c a$. 
5. Prove the identities:
   a) \( \frac{\log_a p}{\log_a q} = \frac{\log_b p}{\log_b q} \)
   b) \( \log y = \log_{\frac{1}{x}} y \)

6. Prove the identity \( \log(x^2 - 1) - \log(x - 1) = \log(x + 1) \) and state the values of \( x \) for which the identity is defined.

7. Which of the following represents an identity?
   A. \( \log x = 100 \)
   B. \( 2\log_3 27 = 6 \)
   C. \( \log x + \log(2x - 3) = \log x \)
   D. \( \log(x^2 - x - 2) - \log(x + 1) = \log(x - 2) \)
8. \( \log x + \log(x + 4) \) is equal to
   A. \( \log(2x + 4) \)
   B. \( \log(x^2 + 4x) \)
   C. \( \log(x^2 + 4) \)
   D. \( \log(x) \log(x + 4) \)

9. \( \log(x^2 - 4) - \log(x - 2) \) is equal to
   A. \( \log(x + 2) \)
   B. \( \log(x^2 - x - 2) \)
   C. \( \log(x - 2) \)
   D. \( \frac{\log(x^2 - 4)}{\log(x - 2)} \)

10. \((\log 2x)^2\) is equal to
    A. \( 2 \log 2x \)
    B. \( \log 4x^2 \)
    C. \( 2 \log 4x \)
    D. \( (\log 2)^2 + 2 \log 2 \log x + (\log x)^2 \)

**Numerical Response** 11. If \( \frac{1}{3} \log \sqrt{x} + \log x^3 = n \log x \) is an identity, then the value of \( n \), to the nearest tenth, is \( \text{_____} \).

**Answer Key**

6. \( x > 1 \)  
7. D  
8. B  
9. A  
10. D  
11. 3.2
Graphing Logarithmic Functions

**a)** The graph of \( y = \log_3 x \) is shown. By using a graphing calculator with window format \( x: [-1, 11, 1] \quad y: [-4, 4, 1] \), sketch on the same grid the logarithmic functions with equations:

(i) \( y = \log_{10} x \)

(ii) \( y = \log_{\frac{1}{3}} x \)

(iii) \( y = \log_{\frac{1}{10}} x \)

**b)** Without using a graphing calculator, make a sketch of the graphs of:

i) \( y = \log_5 x \)  

ii) \( y = \log_{\frac{1}{5}} x \)

**c)** Verify the solution in b) using a graphing calculator.

**d)** Complete the table.

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>x-int</th>
<th>y-int</th>
<th>Asymptote</th>
<th>x-value when y = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \log_3 x )</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>( y = \log_{10} x )</td>
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<tr>
<td>( y = \log_{\frac{1}{3}} x )</td>
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<tr>
<td>( y = \log_{\frac{1}{10}} x )</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( y = \log_b x )</td>
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<td></td>
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</tr>
</tbody>
</table>

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Transformations of Logarithmic Functions

We use the knowledge learned in *Transformations*, to compare the graph of \( y = \log_c x \) to the graph of \( y = a \log_b (x - h) + k \). We use the letter \( c \) to represent the base of the logarithm to distinguish it from the letter \( b \) which is associated with the horizontal stretch.

The graph of \( y = \log_3 x \) is shown.

a) Write the transformation associated with each of the following
   i) \( y = 2 \log_3 x \)
   ii) \( y = \log_3 (x - 2) \)
   iii) \( y = \log_3 x - 2 \)

b) Do any of the above transformations result in a change in the original domain or range?

c) State the equation of the asymptote for each part of a)

Complete Assignment Questions #1 - #10
Assignment

1. Describe how the graphs of the following functions relate to the graph of \( y = \log x \).

   a) \( y = 5 \log x - 2 \)
   
   b) \( \frac{1}{7} y = \log (2x - 3) \)

   c) \( y = \log \frac{1}{3} x + 1 \)
   
   d) \( y = \log (3x + 1) \)

2. Complete the following table:

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \log x )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = 5 \log x + 2 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = 5 \log (x + 2) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = -\log x )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = \log (-x) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = 2 \log 3(x - 6) - 1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = \log(3x - 6) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.a) Complete the following table:

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \log_{4} x )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = \log_{\frac{1}{4}} x )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Describe how the graph of \( y = \log_{4} x \) is related to the graph of \( y = \log_{\frac{1}{4}} x \) in terms of transformations.

c) Hence write \( y = \log_{\frac{1}{4}} x \) in the form \( y = a \log_{4} x \)
4. Consider the graph of the function \( f(x) = a \log_b(x - h) + k \), with \( a, b > 0 \)
Which of the parameters \( a, b, c, h, k \), affect the:

a) domain
b) range
c) asymptote

5. Consider the graph of the function \( f(x) = a \log_b(x - h) + k \), with \( a, b > 0 \).

a) If \( a \) is changed to a negative value, does this affect the domain, range, or asymptote?

b) If \( b \) is changed to a negative value, does this affect the domain, range, or asymptote?

6. Two students, Andy and Holly, were given the following question on an exam.

"A transformation is applied to the graph of \( y = \log x \) and a sketch of both graphs, is shown. Write the equation of the graph which represents the transformed image."

\[
\begin{array}{c}
\begin{array}{c}
\text{y} \\
\text{1}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{x} \\
-1
\end{array}
\end{array}
\]

a) Andy correctly answered the question with the equation of the transformed image as \( y = \log 3x \). Explain from the given sketch how Andy arrived at his solution.

b) Holly also correctly answered the question, but with an equation in the form \( y = \log x + k \). Find the value of \( k \).
7. The x-intercept of the graph of the function \( f(x) = \log_a (x - d) \) is
   A. \( d \)
   B. \( -d \)
   C. \( 1 + d \)
   D. \( 1 - d \)

8. The graph of \( y = \log x \) is transformed to the graph of \( y = \log(2x + 5) \) by a horizontal stretch about the y-axis by a factor of \( p \) followed by a horizontal translation of \( q \) units left. The value of \( p + q \) is
   A. 3
   B. 4.5
   C. 5.5
   D. 7

9. An equation of the asymptote of \( y = 3\log_4(x + 2) - 6 \) is
   A. \( x = -6 \)
   B. \( y = -6 \)
   C. \( x = -2 \)
   D. \( y = -2 \)

10. If the graph of \( y = \log_7 x \) is reflected in the x-axis, the equation of the image can be written in the form \( y = \log_c x \). The value of \( c \), to the nearest hundredth, is _____.

11. The graph of \( y = \log_5 x \) is translated 2 units down. A student writes the equation of the transformed image in the form \( y = \log_5 kx \). The value of \( k \), to the nearest hundredth, is _____.
Answer Key

1. a) vertical expansion about the x-axis by a factor of 5
   vertical translation 2 units down

   b) vertical expansion about the x-axis by a factor of 7
   horizontal compression about the y-axis by a factor of \( \frac{1}{2} \)
   horizontal translation 3 units right

   c) horizontal expansion by a factor of 3
   about the y-axis
   vertical translation 1 unit up

   d) horizontal compression by a factor of \( \frac{1}{3} \)
   about the y-axis
   horizontal translation \( \frac{1}{3} \) unit left

2.

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \log x )</td>
<td>( {x \mid x &gt; 0, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( f(x) = 5 \log x + 2 )</td>
<td>( {x \mid x &gt; 0, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( f(x) = 5 \log (x + 2) )</td>
<td>( {x \mid x &gt; -2, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = -2 )</td>
</tr>
<tr>
<td>( f(x) = - \log x )</td>
<td>( {x \mid x &gt; 0, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( f(x) = \log (-x) )</td>
<td>( {x \mid x &lt; 0, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( f(x) = 2 \log (3x - 6) - 1 )</td>
<td>( {x \mid x &gt; 6, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 6 )</td>
</tr>
<tr>
<td>( f(x) = \log(3x - 6) )</td>
<td>( {x \mid x &gt; 2, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 2 )</td>
</tr>
</tbody>
</table>

3. a)

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Asymptote</th>
</tr>
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<tbody>
<tr>
<td>( f(x) = \log_4 x )</td>
<td>( {x \mid x &gt; 0, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( f(x) = \log_{\frac{1}{4}} x )</td>
<td>( {x \mid x &gt; 0, x \in \mathbb{R} } )</td>
<td>( y \in \mathbb{R} )</td>
<td>( x = 0 )</td>
</tr>
</tbody>
</table>

b) reflection in the x-axis
c) \( y = -\log_4 x \)

4. a) \( h \)  b) none  c) \( h \)

5. a) no  b) domain, and asymptote if \( h \neq 0 \)

6. a) since the x-intercept of the image is \( \frac{1}{3} \) of the original intercept, there is a horizontal compression by a factor of \( \frac{1}{3} \) about the y-axis so therefore Andy is correct by writing \( y = \log 3x \)

b) \( k = \log 3 \)

7. C

8. A

9. C

10. 0.14

11. 0.04
In earlier work we met sequences of numbers. Sequences occur frequently in mathematics. A sequence is a set of numbers in a definite order which are written separated by commas. The terms of a sequence are the values of some function whose domain is the set of natural numbers. A common way to define this function is by a formula which gives the $n^{th}$ term of a sequence.

Write the next two terms of the following sequence and describe a rule which can be used to form the sequence.

a) $1, 3, 5, 7, ...$  
b) $2, 4, 8, 16, ...$  
c) $4000, 2000, 1000, 500, ...$

d) $2, 3, 5, 7, 11, ...$  
e) $1, 1, 2, 3, 5 ...$

The first term of a sequence is written as $t_1$, the second term is written as $t_2$, and the general term, or $n^{th}$ term, is written as $t_n$. The general term provides a formula which can be used to determine any term of a sequence.

For example, the general term of the sequence $1, 3, 5, 7, ...$ is $t_n = 2n - 1$.

The general term to both of these sequences is $t_n = 3(2)^{n-1}$. 

**Infinite Sequence** - A sequence that has an unlimited number of terms eg. $3, 6, 12, ...$

**Finite Sequence** - A sequence that has a specific number of terms eg. $3, 6, 12, ... 192$
a) List the first three terms of the sequence \( t_n = 3n^2 - n \).

b) Determine the eighth term of the sequence \( t_n = 4n^2 - 9 \).

Types of Sequences

In this mathematics program we concentrate on three groups of sequences.

\[\text{Sequences}\]
\[\text{Arithmetic Sequences} \quad \text{Geometric Sequences} \quad \text{Other Sequences}\]

**Arithmetic Sequence** - a sequence where each term is formed from the preceding term by adding a constant (positive or negative) to it, e.g., 7, 11, 15, 19, ...

This constant is called the **common difference**.

The common difference in the example above is _____.

Arithmetic sequences were discussed in earlier math courses.

**Geometric Sequence** - is a sequence where each term is obtained by multiplying the preceding term by a constant, e.g., 2, 6, 18, 54, ...

This constant is called the **common ratio**.

The common ratio in the example above is _____.

**Other Sequences** - sequences which are neither arithmetic nor geometric.
Consider the following sequences from Class Ex. #1. State whether each sequence is arithmetic, geometric, or neither.

- **a)** 1, 3, 5, 7, ...
- **b)** 2, 4, 8, 16, ...
- **c)** 4000, 2000, 1000, 500, ...
- **d)** 2, 3, 5, 7, 11, ...
- **e)** 1, 1, 2, 3, 5 ...

### Geometric Sequences

Geometric sequences will be the main focus of discussion in the remainder of this unit.

### Finding a Common Ratio

We can find the common ratio in a geometric sequence by dividing the second term by the first term or by dividing any term by the previous term.

\[
\frac{t_2}{t_1}, \text{ or } \frac{t_5}{t_4}, \text{ etc., gives the common ratio.}
\]

**common ratio** = \[
\frac{t_n}{t_n - 1}
\]

*not on the formula sheet*

Find the common ratio.

- **a)** 6, 12, 24, ...
- **b)** −1, 5, −25, ...
- **c)** −10, −5, −\(\frac{5}{2}\), ...

**Class Ex. #4**
Developing a Formula to find the General Term

Consider the following sequences

i) \(3, 6, 12, 24, \ldots\)

ii) \(64, -32, 16, -8, \ldots\)

Complete the following:

\[ t_1 = 3 \quad t_1 = \]
\[ t_2 = 6 = 3(2) \quad t_2 = \]
\[ t_3 = 12 = 3(2)^2 \quad t_3 = \]
\[ t_4 = 24 = \quad t_4 = \]
\[ t_5 = \quad t_5 = \]
\[ t_6 = \quad t_6 = \]
\[ t_n = \quad \leftrightarrow \quad \text{General Term} \quad \rightarrow \quad t_n = \]

Consider the general geometric sequence with \(a\) representing the first term and \(r\) representing the common ratio. Express the following in terms of \(a\) and \(r\).

\[ t_1 = \quad t_{39} = \]
\[ t_2 = \quad t_{50} = \]
\[ t_3 = \quad t_{125} = \]
\[ t_4 = \]
\[ t_n = \]

Hence, for a geometric sequence, the general term can be determined using the formula

\[
\boxed{ t_n = ar^{n-1} } \quad \text{on the formula sheet}
\]

where

\(t_n\) is the general term of the geometric sequence
\(a\) is the first term
\(r\) is the common ratio
\(n\) is the position of the term being considered

Apply the following pattern for the terms of a geometric sequence:

\[
t_1, \quad t_2, \quad t_3, \quad t_4, \quad \ldots \quad t_n \quad a, \quad ar, \quad ar^2, \quad ar^3, \quad \ldots \quad ar^{n-1}
\]
Class Ex. #5

Determine the general term and calculate the seventh term for each of the following:

a) 32, 16, 8, ... 

b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, ...$

Class Ex. #6

What is the general term for the sequence $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...$?

Class Ex. #7

Given the sequence 32, 64, 128 ... , 16384, determine:

a) the number of terms in the sequence.

b) which term in the sequence is 1024.

Class Ex. #8

Given a geometric sequence with $t_4 = -54$ and $t_7 = 1458$, find the:

- first term,
- common ratio, and,
- general term.
Class Ex. #9

$x, x + 5, x + 9$ are the first three terms in a geometric sequence. Determine the exact value of each term.

---

**Geometric Means**

**Geometric Means** - are terms which are between two given terms in a geometric sequence. In order to find the geometric means, the two given terms can be thought of as the first and last terms of the sequence.

Class Ex. #10

Insert 4 geometric means between 81 and $\frac{1}{729}$.

---

**Complete Assignment Questions #1 - #15**

**Assignment**

1. For each of the following sequences determine if the sequence is:
   
i) arithmetic, geometric, or neither          ii) finite or infinite

   a) 1, 2, 3                           b) 1, 1, 2, 3, ...             c) 0.5, 0.55, 0.555

   d) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ..., \frac{1}{2187}$           e) $-1, 1, -1, ...$            f) 2, 4, 8, ..., 256

   g) \log a, \log 2a, \log 4a       h) \log a, \log a^2, \log a^4 ...

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2. State the common ratio and find the next two terms of each sequence.
   a) $6, -18, 54, -162, \ldots$
   b) $2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots$
   c) $5, 5\sqrt{3}, 15, \ldots$

3. Find the number of terms in each sequence.
   a) $-6, -12, -24, \ldots, -192$
   b) $512, -256, 128, \ldots, -1$
   c) $-2, 4, -8, \ldots, 1024$

4. Find the position of the last term in each of the following sequences
   a) $2, 14, 98, \ldots, 4802$
   b) $\frac{1}{384}, -\frac{1}{192}, \frac{1}{96}, \ldots, -\frac{16}{3}$

5. Find $a$, $r$, and $\ell_n$ for each geometric sequence given two terms of the sequence.
   a) $t_2 = 28$, $t_5 = 1792$
   b) $t_4 = 64$, $t_8 = \frac{1}{4}$
6. a) In a geometric sequence \( t_2 = 3 \), and \( t_7 = 729 \). Determine \( t_{10} \).

b) In a geometric sequence the third term is 18, and the seventh term is \( \frac{9}{8} \). Determine the exact value of the twelfth term. Explain why there are two possible answers.

7. How many terms are there in a geometric sequence in which there are 7 geometric means?

8. Place three geometric means between 24 and \( \frac{3}{2} \).
9. For the sequence $x - 4, 2x + 4, 9x + 6$ find the value(s) of $x$ and the numerical value of the three terms if the sequence is geometric.

10. Find the $n^{th}$ term for the sequence $2, -6, 18, ...$.

Multiple Choice

A. $t_n = -3(2)^{n-1}$
B. $t_n = 2(-3)^{n-1}$
C. $t_n = (-6)^{n-1}$
D. $t_n = 2(3)^{1-n}$

11. The sequence $\log x, \log x^2, \log x^3, \log x^4, ...$, is

A. geometric with a common ratio of $\log x$
B. geometric with a common ratio of $\log \left(\frac{1}{x}\right)$
C. geometric with a common ratio of $x$
D. not geometric

12. A sequence is defined by $t_n = \frac{1}{2}n - 2, n \in N$. This sequence is

A. finite and geometric
B. finite and not geometric
C. infinite and geometric
D. infinite and not geometric

13. For the geometric sequence $x, 2x + 1, 4x + 10$ the value of $x$ to the nearest hundredth is ______.
14. For a geometric sequence \( t_3 = 4x + 8 \), and \( t_6 = x - 4 \). If the common ratio is \( \frac{1}{2} \), the value of \( t_8 \), to the nearest tenth, is _____.

15. If the sequence \( \log_2 n \), \( \log_4 n \), \( \log_x n \) is geometric, then the value of \( x \), to the nearest tenth, is _____.

---

**Answer Key**

1. a) arithmetic / finite  
   b) neither / infinite  
   c) neither / finite  
   d) geometric / finite  
   e) geometric / infinite  
   f) geometric / finite  
   g) arithmetic / finite  
   h) geometric / infinite

2. a) \( r = -3 \); 486, -1458  
   b) \( r = \frac{2}{3} \); \( \frac{32}{81}, \frac{64}{243} \)  
   c) \( r = \sqrt{3} \); 15\( \sqrt{3} \), 45

3. a) 6  
   b) 10  
   c) 10

4. a) \( t_5 \)  
   b) \( t_{12} \)

5. a) \( a = 7, r = 4, t_n = 7(4^{n-1}) \)  
   b) \( a = 4096, r = \frac{1}{4}, t_n = 47^{-n} \) or \( a = -4096, r = -\frac{1}{4}, t_n = -4096 \left( -\frac{1}{4} \right)^{n-1} \)

6. a) \( a = 1, r = 3, t_{10} = 19683 \)  
   b) \( a = 72, r = \frac{1}{2}, t_n = 72 \left( \frac{1}{2} \right)^{n-1} = 9(2)^{4-n}, t_{12} = \frac{9}{256} \)

   or

   \( a = 72, r = -\frac{1}{2}, t_n = 72 \left( \frac{1}{2} \right)^{n-1} = -9(-2)^{4-n}, t_{12} = -\frac{9}{256} \)

Since \( r^4 = \frac{1}{16} \) has two solutions, there are two sequences and two values for the twelfth term

7. 9  
8. 12, 6, 3 or -12, 6, -3  
9. \( x = -\frac{4}{5} \) or 10; \(-\frac{24}{5}, \frac{12}{5}, -\frac{6}{5} \) or 6, 24, 96

10. B  
11. D  
12. D  
13. 0.17  
14. 1.5  
15. 16.0

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**Geometric Series and Applications of Exponential and Logarithmic Functions Lesson #2: Geometric Series**

**Series**

Series - the terms of a sequence are added together to form a series.

For example, 2, 4, 8, 16, ... is a geometric sequence

\[ 2 + 4 + 8 + 16 + ... \] is a geometric series

**The Formula for the Sum of n Terms of a Geometric Series**

\[ S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-2} + ar^{n-1} \]

\[ rS_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n \]

Subtract line 2 from line 1 \[ S_n - rS_n = a - ar^n \]

\[ S_n(1 - r) = a(1 - r^n) \]

\[ S_n = \frac{a(1 - r^n)}{1 - r}, \quad r \neq 1 \]

on the formula sheet

Since \( l \) (the last term) = \( ar^{n-1} \), the above formula can be written

\[ S_n = \frac{a - rl}{1 - r}, \quad r \neq 1 \]

on the formula sheet

When \( r = 1 \) the series is \( a + a + a + ... = na \)

Note that the formula could also be written \( S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1 \)

**Class Ex. #1**

Determine the sum of the first seven terms of the sequence \(-3, 6, -12, ...\).

**Class Ex. #2**

Find the sum of \( 4 - 12 + 36 - ... - 8748 \).
The sum of a certain number of terms in the series \((-2) + 8 + (-32) + \ldots\) is \(-104,858\). What is the last term that would make this series add up to \(-104,858\)?

In a geometric sequence the fifth term is 1024 and the common ratio is 4. Find the sum of the first seven terms of this series.

A golf ball is dropped from the top of a building 100 m above a paved road. In each bounce the ball reaches a vertical height that is \(\frac{3}{4}\) the previous vertical height. Determine:

a) the vertical height (to the nearest tenth of a metre) of the ball after the seventh bounce.

b) the total vertical distance (to the nearest tenth) travelled by the ball when it contacts the floor for the seventh time.
c) How many times does the ball need to bounce to travel approximately 675 m in vertical distance?

Consider the geometric series defined by $S_n = 5(3^n - 1)$.

a) Find the first four terms of the geometric series defined by $S_n = 5(3^n - 1)$.

b) Find $t_9$ without using the formula $t_n = ar^{n-1}$

c) Find an expression for $t_n$ by two different methods.
To find \( t_n \) in a series defined in terms of \( S_n \), use the formula

\[
t_n = S_n - S_{n-1}
\]

For example, \( t_8 = S_8 - S_7 \), or \( t_4 = S_4 - S_3 \)

**Complete Assignment Questions #1 - #13**

**Assignment**

1. Find the sum of each series. *Answer as an exact value unless otherwise indicated.*

   a) \( 4 + 16 + 64 + \ldots \) (\( S_8 \))

   b) \( 12 + 6 + 3 + \ldots \) to 6 terms

   c) \( 64 - 32 + 16 - \ldots \) (\( S_9 \))

   d) \( \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \ldots \) (\( S_7 \))

   e) \( 1 + 3 + 9 + \ldots + 729 \)

   f) \( 512 + (-256) + 128 + \ldots + (-1) \)

   g) \( -\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \ldots \) (\( S_{11} \))

   (to the nearest tenth)

   h) \( \frac{1}{384} - \frac{1}{192} + \frac{1}{96} - \ldots - \frac{16}{3} \)

   (to the nearest tenth)
2. Find the exact sum of the first eight terms of a geometric series that has a general term of \( t_n = -3(-2)^{1-n} \).

3. How many terms are required in the series \((-6) + (-12) + (-24) \ldots\) to add to a sum of \(-378\)?

4. For the geometric series \(125 + (-25) + 5 + \ldots\) find:
   a) the sixth term
   b) the sum of six terms

5. A line is divided into 5 parts whose lengths form a geometric sequence. If the shortest length is 2 cm, and the longest 162 cm, find the length of the whole line.
6. Find the exact sum of the series $5 + 3 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{512}$.

7. Find $t_n$ if $S_n = \frac{75}{4}(5^n - 1)$.

8. A mother, during a hectic day, was convinced by her son that he should have a weekly allowance that is increased every two weeks. In weeks one and two he receives 1 cent per week, in weeks three and four he receives 2 cents per week. If his allowance continues to double every two weeks, how much would his allowance be in week 30? What is the total amount he received in allowance during the thirty weeks?
9. Legend has it that long ago a king was so pleased with the game of chess that he decided to reward the inventor of the game, Sessa, with whatever he wanted. Sessa, taking the request seriously, asked for a resource instead of money. Specifically, he asked for one grain of wheat for the first square of a chessboard, two grains of wheat for the second square, four grains for the third square, and so on until the entire chess board was full (there are 64 squares on a chess board) Find an expression for:

a) the amount of grain on the last square of the board.

b) the total amount of grain needed to fulfill Sessa’s request.
(Don’t be surprised at how big this is - this amount is said to represent many times the world’s annual crop of wheat!)

10. The first term of a geometric sequence is 3. The sum of the first two terms of a corresponding series is 15 and the sum of the first 3 terms of the series is 63. The common ratio is

A. 3
B. 4
C. 5
D. \(\frac{63}{15}\)

11. The general term of a sequence is \(t_n = 3(2)^{n-1}\), \(n \geq 1\). The sum of the first seven terms of the corresponding series, to the nearest whole number, is _____.
12. A hard rubber ball is dropped from a moving truck with a height of 5 m. The ball rises \( \frac{4}{5} \) of the height from which it fell after each bounce. The total vertical distance the ball has travelled at the moment it hits the ground for the eighth time, to the nearest tenth of a metre, is _____.

13. If \( S_n = 2(3^n - 1) \) represents the sum of a geometric series, then the value of the seventh term, \( t_7 \), to the nearest whole number, is _____.

**Answer Key**

1. a) 87 380  
   b) \( \frac{189}{8} \)  
   c) \( \frac{171}{4} \)  
   d) \( \frac{127}{8} \)  
   e) 1093  
   f) 341  
   g) -3.5  
   h) -3.6

2. -\( \frac{255}{128} \)  
3. 6  
4. a) -\( \frac{1}{25} \)  
   b) \( \frac{2604}{25} \)  
5. 242 cm  
6. 8 \( \frac{511}{512} \) or \( \frac{4607}{512} \)

7. \( t_n = 75(5^n - 1) \) or \( 3(5^n + 1) \)

8. $163.84, \quad$655.34

9. a) \( 9.22 \times 10^{18} \) grains  
   b) \( 1.84 \times 10^{19} \) grains

10. B  
11. 381  
12. 36.6  
13. 2916
Investigating an Infinite Geometric Series with \( r = \frac{1}{2} \)

David buys a pizza. He eats half of the pizza, then half of the remaining piece, then half of the remaining piece, etc., etc. Continuing in this way, it looks like he will never finish the pizza, because there will always be half of a piece left.

a) Explain why in practice this could not happen.

b) Represent this situation with a geometric series whose first term is \( \frac{1}{2} \).

c) For this series, determine the values of \( a \) and \( r \), and find a formula for \( S_n \).

d) Use the table feature of a graphing calculator to complete the table to three decimal places.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

e) Plot the values in d) on the grid. Note that \( n \) is defined only on the set of natural numbers, so do not join the points.

f) It would appear from the grid that, as \( n \) gets larger, the sequence of sums \( S_1, S_2, S_3, \ldots, S_n \) gets closer and closer to the value _____.

This kind of series is called **convergent** as the sequence of sums converges closer and closer to a particular value. It would appear that if we added the terms of the series indefinitely, we will get closer and closer to the value 1.

We say that the sum of the infinite geometric series \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \) is 1, or that the sum to infinity of the series is 1.

We use the symbol \( S \) or \( S_\infty \) to represent the sum of the infinite series.

Note that the sum of a finite number of terms of this series will never actually reach 1.

g) Explain with reference to the pizza why the sum of this infinite series should be 1.
Investigating an Infinite Geometric Series with $r = 2$

Consider the infinite geometric series $2 + 4 + 8 + 16 + \ldots$

a) For this series, determine the values of $a$ and $r$, and find a formula for $S_n$.

b) Use the table feature of a graphing calculator to complete the table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In this case, as $n$ gets larger, the sequence of sums $S_1, S_2, S_3, \ldots S_n, \ldots$ gets larger and larger and does not converge to a finite value.

This kind of series is called **divergent** and does not have a sum to infinity.

We have seen that an infinite geometric series with $r = \frac{1}{2}$ converges to a particular value called the sum to infinity and that an infinite geometric series with $r = 2$ diverges and does not have a sum to infinity.
**Warm-Up #3  Investigating Values of \( r^n \) as \( n \) Approaches Infinity**

In this investigation, we can choose:

- any value of \( r \) less than \(-1\)
- any value of \( r \) between \(-1\) and 0
- any value of \( r \) between 0 and 1
- any value of \( r \) greater than 1.

We have suggested the values \(-2, -\frac{1}{4}, \frac{2}{3}, \text{ and } 3\), but any values in the range will do.

**a)** Complete the table.

<table>
<thead>
<tr>
<th></th>
<th>( r &lt; -1 )</th>
<th>(-1 &lt; r &lt; 0 )</th>
<th>( 0 &lt; r &lt; 1 )</th>
<th>( r &gt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( eg )</td>
<td>( r = -2 )</td>
<td>( r = -\frac{1}{4} )</td>
<td>( r = \frac{2}{3} )</td>
<td>( r = 3 )</td>
</tr>
<tr>
<td>( n )</td>
<td>( r^n )</td>
<td>( r^n )</td>
<td>( r^n )</td>
<td>( r^n )</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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<tr>
<td>4</td>
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</tr>
<tr>
<td>10</td>
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<td></td>
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</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**b)** Complete the following statements based on your observations in a).

As \( n \) gets larger and larger:

- the sequence is convergent and approaches the value 0 if \( r = \ldots \), or \( r = \ldots \).
- the sequence is divergent if \( r = \ldots \), or \( r = \ldots \).

As \( n \) gets larger and larger:

- \( r^n \) gets closer and closer to 0, provided \(-1 < r < 1\), i.e. \(|r| < 1\).
- \( r^n \) gets larger and larger if \( r < -1 \) or \( r > 1 \), i.e. \(|r| > 1\).

We can use this rule to determine a formula for the sum of an infinite geometric series.
The Formula for the Sum of an Infinite Geometric Series

Consider the infinite series $S = a + ar + ar^2 + ar^3 + \ldots$

The sum of $n$ terms of this series is given by the formula $S_n = \frac{a(1 - r^n)}{1 - r}$, $r \neq 1$.

Provided $-1 < r < 1$, $r^n$ will get closer and closer to zero as $n$ gets closer and closer to infinity.

This means that in determining the formula for the sum of an infinite series, $S$, we can replace $r^n$ by zero in the formula for $S_n$.

This gives $S = \frac{a}{1 - r}$, provided $|r| < 1$

If $|r| \geq 1$, the sum of an infinite geometric series is not defined.

\[ S = \frac{a}{1 - r}, \quad |r| < 1 \] on the formula sheet

Overview

The following flowchart summarizes geometric series learned to this point

**Geometric Series**

**Finite Geometric Series**
A finite geometric series has a sum which can be determined using

\[ S_n = \frac{a(1 - r^n)}{1 - r}, \quad r \neq 1 \]

or

\[ S_n = \frac{a - ar}{1 - r}, \quad r \neq 1 \]

**Infinite Geometric Series**

**Convergent**
An infinite series is convergent, (i.e. the series approaches a particular value which we say is the sum of that series) if $-1 < r < 1$ or $|r| < 1$

The formula used to find the sum of an infinite geometric series which is convergent is

\[ S = \frac{a}{1 - r} \]

**Divergent**
The infinite series is divergent (i.e. the series does not approach a particular value) if $|r| > 1$
Find the common ratio for each of the following geometric series and state whether a sum to infinity exists. Find this sum where it exists.

a) \(1 + \frac{1}{3} + \frac{1}{9} + \ldots\)  
b) \(1 - 5 + 25 - \ldots\)  
c) \(2 - 1 + \frac{1}{2} - \ldots\)

The first term of a geometric series is 2 and the sum to infinity is 4. Find the common ratio.

a) Write the repeating decimal 0.0\(\overline{7}\) as an infinite geometric series.

b) Find the sum to infinity of the series in a) and hence write 0.0\(\overline{7}\) as a fraction.

**Assignment**

1. Find the common ratio for each of the following geometric series and state whether a sum to infinity exists. Find this sum where it exists.

   a) \(4 + 2 + 1 + \ldots\)  
   b) \(5 - 1 + \frac{1}{5} - \ldots\)  
   c) \(-4 + 6 - 9 + \ldots\)
d) $1 + 1 + 1 + \ldots$

e) $10 - 9 + 8.1 - \ldots$

f) $1 - 1 + 1 - \ldots$

g) $15 - 9 + \frac{27}{5} - \ldots$

h) $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \ldots$

i) $0.0001 + 0.001 + 0.01 + \ldots$

j) $2^6 + 2^5 + 2^4 + \ldots$

k) $2^4 + 2^5 + 2^6 + \ldots$

l) $\frac{3}{100} + \frac{3}{10000} + \frac{3}{1000000} + \ldots$

m) $3 + \sqrt{3} + 1 + \frac{1}{\sqrt{3}} + \ldots$
2. Consider the geometric sequence $12 + 6 + 3 + ...$.
   a) Find the sums for 10 and 12 terms to four decimal places.

   b) Explain why these sums are almost equal.

   c) Predict the sum for the infinite series.

   d) Calculate the sum for the infinite series.

3. The first term of a geometric series is 81 and the third term is 1. Find the sum to infinity of each of the two possible series.

4. a) Show that $4x^3$, $8x^{-3}$, and $16x^{-2}$ could be the first three terms of a geometric series.

   b) If $x = 8$, show that the sum to infinity exists, and find the infinite sum.
5. The infinite geometric series is given by \(1 - 3x + 9x^2 - 27x^3 + \ldots\).

If the infinite sum is \(\frac{4}{9}\), determine the numerical value of the common ratio.

6. Use an infinite series to express the following repeating decimals as fractions.

   a) \(0.\overline{5}\)  
   b) \(0.\overline{35}\)  
   c) \(0.3\overline{5}\)

7. During the first week of operation, an oil well produced 8000 barrels of oil. The production dropped by 2% each week.

   a) Calculate to the nearest barrel:
      i) The number of barrels produced in week 6.
      
      ii) The total number of barrels produced in the first ten weeks of production.
      
      iii) The total number of barrels which could be produced before the well runs dry.

   b) Why might the actual number of barrels produced differ from the answer to a) iii)?
8. The sum of the infinite geometric series \( t + t^2 + t^3 + t^4 + \ldots \) is \( 4t \), \( t \neq 0 \). The value of \( t \) is

A. \( \frac{4}{3} \)

B. \( \frac{3}{4} \)

C. \( \frac{1}{2} \)

D. \( \frac{1}{4} \)

9. An expression for the sum of the infinite series \( x^2 + 1 + \frac{1}{x^2} + \frac{1}{x^4} + \ldots \) is

A. \( \frac{x^4}{x^2 - 1} \)

B. \( \frac{x^4}{x^2 + 1} \)

C. \( \frac{1}{x^2 - 1} \)

D. \( x^4 \)

10. The common ratio of a geometric series is \( -\frac{2}{3} \) and the sum to infinity is \(-12\). The second term, to the nearest tenth, is ______.
11. The third term of a geometric series is $\frac{4}{3}$, and the sixth term is $\frac{32}{81}$. The difference between the sum of the first five terms and the sum to infinity of the series, correct to the nearest tenth, is _____. 

**Answer Key**

1. a) $r = \frac{1}{2}$, $s = 8$  
   b) $r = -\frac{1}{2}$, $s = \frac{25}{6}$  
   c) $r = -\frac{3}{2}$, $s$ does not exist  
   d) $r = 1$, $s$ does not exist  
   e) $r = -\frac{9}{10}$, $s = \frac{100}{19}$  
   f) $r = -1$, $s$ does not exist  
   g) $r = -\frac{3}{5}$, $s = \frac{75}{8}$  
   h) $r = -\frac{1}{2}$, $s = \frac{1}{2}$  
   i) $r = 10$, $s$ does not exist  
   j) $r = \frac{1}{2}$, $s = 128$  
   k) $r = 2$, $s$ does not exist  
   l) $r = \frac{1}{100}$, $s = \frac{1}{33}$  
   m) $r = \frac{1}{\sqrt{3}}$, $s = \frac{3\sqrt{3}}{\sqrt{3} - 1}$ or $\frac{9 + 3\sqrt{3}}{2}$

2. a) $S_{10} = 23.9766$, $S_{12} = 23.9941$  
   b) terms 11 and 12 are very small  
   c) 24  
   d) 24

3. $S = \frac{729}{8}$ or $\frac{729}{10}$  

4.a) $\frac{t_3}{t_2} = \frac{t_2}{t_1} = 2x^\frac{5}{7}$. Common ratio so terms can form a geometric series.  
   b) $S$ exists since $r = \frac{1}{16}$. $S = \frac{1024}{15}$

5. $r = \frac{5}{4}$

6. a) $\frac{5}{9}$  
   b) $\frac{35}{99}$  
   c) $\frac{16}{45}$

7. a) i) 7231  
   ii) 73171  
   iii) 400 000  
   b) Once the number of barrels produced per week drops below a certain level, it becomes uneconomical to keep the well open.

8. B

9. A

10. 13.3

11. 1.2

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Warm-Up

Consider the geometric series $2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$. In this lesson we introduce a new notation which enables us to write the series in an abbreviated form.

Sigma Notation

In the Greek alphabet the letter Σ (sigma) corresponds to the English letter S. We use sigma notation to represent the sum of a series.

The example below represents the geometric series $2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$

$$
\sum_{k=1}^{8} 2^k
$$

8 is the upper limit of summation
(i.e. continue summing until $k = 8$)

this symbol is
the Greek letter
Sigma and it
stands for sum

8
$$\sum_{k=1}^{8} 2^k$$

the general term of the
 corresponding sequence

1 is the lower limit of summation
(i.e. begin summing with $k = 1$)

Read as the summation of $2^k$ from 1 to 8.

Note! $k$ is a natural number. The choice of $k$ as a variable is arbitrary. In some texts, the letters $i$ or $n$ are often used.

Class Ex. #1

Write the series represented by $\sum_{k=1}^{8} 2^k$ in expanded form, and determine the sum. How many terms are there in this series?
a) Write the series represented by $\sum_{k=5}^{8} 5(2^k + 1)$ in expanded form, and determine the sum.

b) How many terms are there in this series?

c) Describe the relationship between the upper limit, the lower limit and the number of terms in the series.

The number of terms = (upper limit − lower limit) + 1

Write the series $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32}$ in sigma notation.

Determine the value of $\sum_{k=0}^{\infty} 5 \left( \frac{1}{3} \right)^k$
A sequence is defined by the recursive formula \[ \begin{align*}
  t_1 &= 3 \\
  t_n &= 5t_{n-1}, \quad n > 1
\end{align*} \]

a) List the first four terms of the sequence.

b) Find the sum of the first 12 terms of the corresponding series.

c) Represent the series in b) using sigma notation.

Complete Assignment Questions #1 - #10

Assignment

1. State the number of terms in each series.

   a) \[ \sum_{n=1}^{6} 5n \]   
   b) \[ \sum_{z=17}^{32} 2^{-z} \]   
   c) \[ \sum_{k=a}^{a+9} 2n - 5 \]

2. For each of the following series:

   i) write the series in expanded form
   ii) determine whether the series is geometric or not
   iii) find the sum of the series

   a) \[ \sum_{n=3}^{8} 3(2^n - 1) \]   
   b) \[ \sum_{r=4}^{7} (3r - 2) \]
3. Write each series in summation notation.
   
a) $3 + 9 + 27 + 81 + 243 + 729$
   
b) $5 - 5n + 5n^2 - 5n^3 + 5n^4$
   
c) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{4096}$
   
d) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ...$

4. a) Find the first five partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$.
b) Use your answers to a) to estimate \[ \sum_{n=1}^{\infty} \frac{1}{2^n + 1}. \]

c) Determine the exact value of \[ \sum_{n=1}^{\infty} \frac{1}{2^n + 1}. \]

5. Find the value of \[ \sum_{n=2}^{5} \frac{-2^n}{3}. \]

6. A sequence is defined by the recursive formula \[ \begin{cases} t_1 = 5 \\ t_{n+1} = 0.2t_n, \quad n \geq 1 \end{cases} \]

a) List the first four terms of the sequence.

b) Find the sum of the first 10 terms of the corresponding series to the nearest hundredth.

c) Represent the series in b) using sigma notation.

7. A series is defined by the recursive formula \[ \begin{cases} t_1 = -3 \\ t_n = -\frac{1}{3}t_{n-1}, \quad n > 1 \end{cases} \]

a) Find the sum to infinity of the series.

b) Represent this series using sigma notation.
Multiple Choice

8. The value of \( \sum_{n=1}^{4} 5^{\log_5 n} \) is

A. \( \frac{\log 24}{\log 5} \)

B. 10

C. 24

D. 780

Numerical Response

9. If there are 29 terms in the series \( \sum_{k=n+2}^{2n} 3^k \), then the value of \( n \) is _____.

10. The value of \( \sum_{i=2}^{6} \log_i i^3 \), to the nearest whole number, is _____.

Answer Key

1. a) 6  b) 16  c) 10

2. a) i) 12 + 24 + 48 + 96 + 192 + 384  
      ii) geometric  
      iii) 756  
      b) i) 10 + 13 + 16 + 19  
      ii) not geometric  
      iii) 58  
      c) i) \( \log_{24}1 + \log_{24}2 + \log_{24}3 + \log_{24}4 \)  
      ii) not geometric  
      iii) 1  
      d) i) 0.15 + 0.015 + 0.0015 + ...  
      ii) geometric  
      iii) \( \frac{1}{6} \)

3. a) \( \sum_{n=1}^{6} 3^n \)  
   b) \( \sum_{k=1}^{5} 5(-n)^{k-1} \)  
   c) \( \sum_{n=1}^{12} \left(\frac{1}{2}\right)^n \)  
   d) \( \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} \)

4. a) \( S_1 = \frac{1}{4}, S_2 = \frac{3}{8}, S_3 = \frac{7}{16}, S_4 = \frac{15}{32}, S_5 = \frac{31}{64} \)  
   b) \( \frac{1}{2} \)  
   c) \( \frac{1}{2} \)

5. -20

6. a) 5, 1, 0.2, 0.04  
   b) 6.25  
   c) \( \sum_{n=1}^{10} 5(0.2)^{n-1} \)

7. a) \( -\frac{9}{4} \)  
   b) \( \sum_{n=1}^{\infty} (-3)^{2-n} \)

8. B  

9. 30  

10. 15

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In earlier math courses, we learned how to calculate compound interest. In this lesson we will relate compound interest, geometric sequences, and exponential functions and show that the compound interest formula is actually an example of the general term of a geometric sequence.

**Compound Interest**

In simple interest the principal at the beginning of the second year is the same as the principal at the beginning of the first year.

In compound interest the interest earned during the first year is added to the original principal to form a different principal.

**Exploration**

A bank offers two types of savings bond:

- Regular Savings Bond which pays simple interest at 9% per year
- Compound Savings Bond which pays interest at 9% per year compounded annually.

**Simple Interest**

a) The simple interest each year

   is 9% of $5000 = ______.

b) Find the value of the bond at the end of each of the first 8 years, and complete the table.

<table>
<thead>
<tr>
<th>End of Year</th>
<th>Amount($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5450</td>
</tr>
<tr>
<td>2</td>
<td>5900</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
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<td>5</td>
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<tr>
<td>6</td>
<td></td>
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<tr>
<td>7</td>
<td></td>
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<tr>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>
**Compound Interest**

c) Complete the following to determine the compound interest and the value of the bond.

**End of Year 1:** Value of Bond = Principal + Interest

\[
\text{End of Year 1: Value of Bond} = 5000 + 5000(0.09) = 5000(1 + 0.09) = 5000(1.09)
\]

**End of Year 2:** Value of Bond = Principal + Interest

\[
\text{End of Year 2: Value of Bond} = 5000(1.09) + 5000(1.09)(0.09) = 5000(1.09)(1 + 0.09) = 5000(1.09)(1.09) = 5000(1.09)^2
\]

**End of Year 3:** Value of Bond = Principal + Interest

\[
\text{End of Year 3: Value of Bond} = 5000(1.09)^2
\]

d) The value at the end of each year is 1.09 times the value at the end of the previous year. This defines a geometric sequence with common ratio \(r = 1.09\). Complete the table.

<table>
<thead>
<tr>
<th>Term</th>
<th>End of Year</th>
<th>Value of Bond</th>
<th>Amount ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_1)</td>
<td>1</td>
<td>5000(1.09)</td>
<td>5450</td>
</tr>
<tr>
<td>(t_2)</td>
<td>2</td>
<td>5000(1.09)^2</td>
<td></td>
</tr>
<tr>
<td>(t_3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t_4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t_5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t_6)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t_7)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t_8)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t_n)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The general term formula for a geometric sequence is
\[
t_n = ar^{n-1} = 5000(1.09)(1.09)^{n-1} = 5000(1.09)^n.
\]
e) In general, if $A$ = the final amount (or the value of the bond), $P$ = the initial principal, $i$ = the annual interest rate, and $n$ = the number of years, we have the formula:

$$A = \ldots$$

f) Plot the data from the simple interest and compound interest tables. Do not join the points.

- The graph shows that the simple interest bond is growing in a \textbf{linear} pattern and the compound interest bond is growing more quickly, or \textbf{exponentially}.

- This example shows the connection between geometric sequences and exponential functions where the domain is the set of natural numbers.

g) Write the formula for the exponential function in this example in the form $y = ab^n$, $n \in N$. 
In the previous explorations, interest is compounded on an annual basis. In practice, compounding can take place over any period of time, e.g., semi-annually, monthly, daily, continuously, etc.

**Compound Interest Formula**

The formula which can be used to calculate compound interest is

\[ A = P(1 + i)^n \]

where,
- \( A \) represents the final amount
- \( P \) represents the initial principal
- \( i \) represents the interest rate per compounding period
- \( n \) represents the number of compounding periods.

- Note that \( i \) does NOT always represent the annual interest rate.
- Note that \( n \) does NOT always represent the number of years.

Class Ex. #1

$7000 is invested in a 6 year GIC compounded quarterly at a rate of 5% per annum. Determine the value of the investment at the end of the term.

\[ A = \]
\[ P = \]
\[ i = \]
\[ n = \]

Class Ex. #2

Christine invested $2500 for 4 years compounded semi-annually and received $843.26 interest. What was the annual rate of interest?

\[ A = \]
\[ P = \]
\[ i = \]
\[ n = \]

Class Ex. #3

Barbara invests $8000 in an account which pays compound interest of 6% per annum compounded monthly. How long would it take, to the nearest necessary month, for her investment to double in value?

\[ A = \]
\[ P = \]
\[ i = \]
\[ n = \]

Complete Assignment Questions #1 - #8
Assignment

1. Bobby invested $2000 in Canada Savings Bonds @ 8%/year. Calculate the amount of money he will receive in 5 years if the money is:
   a) compounded annually for 5 years
   b) compounded semi-annually for 5 years

2. The value of an investment is given by \( f(x) = 2500(1.062)^x \), where \( x \) is the number of years for which the investment is held. Determine the number of complete years until the investment is worth at least $6000.

3. If a student’s summer job savings of $3000 is invested at 12% per year compounded monthly, how many months will it take to earn at least $600 in interest?

4. Find the interest rate per annum (to one decimal place) at which a $3000 investment, compounded quarterly, will double in value over a period of 5 years.
5. Jane would like to visit her family in Australia. She figures that she will need a total of $9000 for the plane ticket and expenses. She goes to a local bank in Calgary to open a savings account. After some thought she decides to deposit $7500 into an account that pays interest at 8%/a compounded quarterly. How many compounding periods will it take her to reach her goal?

6. Mary-Ann invested $3500 in a Canada Savings Bond at an interest rate of 5.4% per year compounded monthly. Carlos invested $3000 in a G.I.C. at an interest rate of 6.8% per year, compounded annually. After how many years will the two investments be approximately equal in value?

7. A student invests $100 @ 8% per year compounded semi-annually. The amount of money that the student will have at the end of each year is increased from the amount at the end of the previous year by a factor of
   
   A. 1.04  
   B. 1.08  
   C. 1.0816  
   D. 1.16

8. George invests $2500 in an account which pays compound interest of 8.1% per annum compounded quarterly. The number of quarters it will take George’s investment to at least double in value is _______.

**Answer Key**

1. a) $2938.66  b) $2960.49
2. 15
3. 19
4. 14.1%
5. 10
6. 13
7. C
8. 35

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Absolute Value Publications
Geometric Series and Applications of Exponential and Logarithmic Functions Lesson #6: Applications of Exponential Functions

**Warm-Up Review**

Recall the following

- **An exponential function** is a function whose equation is of the form
  \[ y = ab^x \]  where \( a \neq 0, b > 0, x \in \mathbb{R} \)

- For \( a > 0, \)
  - when \( b > 1, \) the function represents a growth function.
  - when \( 0 < b < 1, \) the function represents a decay function.

**Writing an Equation using** \( y = ab^x \)

There are many applications of exponential functions in real life. In some cases, the function \( y = ab^x \) can be written in “disguised” form; e.g. \( A = P(1 + i)^n \) is an example of an exponential function whose base is \( 1 + i \). In this lesson, we will meet further real life applications of exponential growth and decay.

We can use variations of the formula \( y = ab^x \) (such as \( A = P(1 + i)^n \)) to solve problems involving population growth, growth of bacteria, radioactive decay etc.

---

**Class Ex. #1**

In 2002 the university population of a country was 160 000 increasing at an annual rate of 4.5%.

a) Write an equation to represent the university population \( (P) \) of the country as a function of the number \( (n) \) of years since 2002.

b) Determine the university population in the year 2005.

c) If the population continues to grow at this rate, determine the number of years, to the nearest year, for the population to double the number in 2002.
The number of fish in a lake is decreasing by 5% each year as a result of overfishing.

a) Write an equation to represent the number of fish present after $t$ years. Use $N_0$ to represent the initial population and $N(t)$ to represent the final population.

b) If there were 2500 fish present in January 2000, how many would you expect to be present in January 2005?

c) How many years, to the nearest tenth, would it take for the fish population to reduce to half of the number in January 2000?

---

**Continuous Growth and Decay**

The university population in Class Example #1 increased by 4.5% per year. This increase probably occurred at the beginning of each semester and did not occur continuously throughout the year. Other populations, such as the population of the world, increase at a continuous rate. The formula for continuous growth is

$$P = P_0e^{kt}$$

or as a function $P(t) = P_0e^{kt}$

where $P_0$ = the initial population

$P(t)$ = the population at time $t$

$k$ = the growth constant ($k > 0$)

The formula for continuous decay is

$$P = P_0e^{-kt}$$

or as a function $P(t) = P_0e^{-kt}$, $k > 0$

---

The intensity, $I_0$, of a light source is reduced to $I$ after passing through $d$ metres of a fog according to the formula $I = I_0e^{-0.12d}$. In what distance, to the nearest hundredth of a metre, will the intensity be reduced to one quarter of its original value?
Developing a Formula for Doubling Time, Half-Life, etc.

The value of a bond, in dollars, is represented by \( V = 1200(1.0595)^t \) where \( t \) is the time in years.

a) State the initial value of the bond.

b) Determine the value, to the nearest dollar, after six years.

c) Determine the time taken, to the nearest year, for the bond to double in value.

d) The equation for \( V \) can be written as an exponential function with base 2 in the form

\[
V = 1200(2)^p,
\]

where \( p \) is the time taken for the value to double.

Use this equation to determine the value of the bond, to the nearest dollar, after six years and compare your answer with b).

Formula

Class Example #4 is an example of the following general formula which can be used for solving problems involving doubling time, tripling time, half-life, etc.

\[
A_t = A_0 C^\frac{t}{p}, \quad \text{or as a function} \quad A(t) = A_0 C^\frac{t}{p}
\]

where,

\[
\begin{align*}
A_0 & = \text{initial amount} \\
A_t & = \text{amount at time } t \\
C & = \text{a constant, eg. } 2 \text{ for doubling, } 3 \text{ for tripling, } \frac{1}{2} \text{ for half life} \\
t & = \text{time} \\
p & = \text{period of time for doubling, tripling, halving, etc.}
\end{align*}
\]
A patient feeling ill had a sample of bacteria taken from her throat. The sample contained 387 cells. Twenty-four hours later the sample was recounted and was found to contain 8012 cells. Find the doubling period of the bacteria to the nearest tenth of an hour.

A radioactive isotope has a half-life of 7 years.

a) Use the formula \( A_t = A_0 C^t \) with \( C = \frac{1}{2} \) to determine how much of the isotope must initially be present to decay to 60 grams in 14 years.

b) Write an equivalent formula with \( C = 2 \) which would solve the problem in a).

c) In this particular example, explain how the solution to the problem in a) could be found without using any formula.

In April 1986, the nuclear accident at Chernobyl contaminated the atmosphere with quantities of radioactive iodine-131. If the half-life of radioactive iodine-131 is 8.1 days, determine the number of days, to the nearest day, it took for the level of radiation to reduce to 2% of the original level.
Assignment

1. The number $N$, of throat swab bacteria being grown in a culture after $t$ hours, is given by the formula $N = N_0(10^{0.43t})$, where $N_0$ is the original number of bacteria. Find the number of bacteria present after 8 hours if there are initially 500 bacteria in the culture.

2. A sports car priced at $60 000 depreciates at a rate of 14% per year. The value after $n$ years is given by $A = 60 000(0.86)^n$ where $A$ is the amount after depreciation. How long, to the nearest year, will it take for the value of the car to depreciate to $18 000$?

3. The value of a type of robotic technology depreciates 25% per year.
   a) Write an exponential function to represent the value of this robotic technology after $t$ years.
   b) How many years, to the nearest whole year, would it take for the value of a robot which initially cost $575 000 to depreciate to a value of $25 000$?

4. A high school population in a large city is growing at a rate of 3.5% per annum. If the high school population continues to grow at this rate, it is projected that the population will reach 2000 in 5 years. What is the current high school population?
5. A quantity of water contains 500 g of pollutants. Each time the water passes through a filter, 18% of the pollutants are removed. How many filters are needed to reduce the mass of pollutants to less than 150 g?

6. An x-ray beam of intensity, \( I_0 \), in passing through absorbing material \( x \) millimeters thick merges with an intensity, \( I \), given by \( I = I_0 e^{-kx} \). When the material is 9 millimetres thick, 50% of the intensity is lost.

   a) Calculate the value of the constant \( k \) to the three decimal places.

   b) What percentage intensity, to one decimal place, remains if the material is 20 millimetres thick?

7. The population of a South American country is 40 million. Assuming the population is growing continuously, the formula \( P = 40e^{0.014t} \) determines the population, in millions in \( t \) years time. Calculate the population, to the nearest tenth of a million, fifteen years from now.
8. A hot piece of metal loses heat according to the formula \( T = T_0 e^{-0.2t} \), where \( T \) is the temperature difference between the metal and the surrounding air after \( t \) minutes and \( T_0 \) is the initial temperature difference.

a) If the initial temperature of the metal was 330°C and of the air 30°C, find the temperature of the metal, to the nearest degree, after 5 minutes.

b) A different piece of hot metal cools to a temperature of 200°C after 8 minutes. What was the original temperature of the metal, to the nearest degree, if the air temperature was 27°C?

9. How much of a radioactive substance must be present to decay to 30 grams in 12 years if the half–life of the substance is 5.2 years? Round answer to the nearest gram.

10. A lab technician placed a bacterial cell into a vial at 5 a.m. The cells divide in such a way that the number of cells doubles every 4 minutes. The vial is full one hour later.

a) How long does it take for the cells to divide to 4096?

b) At what time is the vial half full?

c) At what time is the vial \( \frac{1}{16} \) full?
11. A radioactive isotope has a half–life of approximately 45 minutes. How long would it take for 480 mg of the isotope to decay to 15 mg?

12. The population of germs in a dirty bathtub doubles every 20 minutes. How long, to the nearest minute, would it take for the population to triple?

13. A radioactive isotope has a half–life of approximately 25 weeks. How much of a sample of 50 grams of the isotope would remain after 630 days? (Round answer to the nearest hundredth of a gram.)

14. What is the half–life, to the nearest month, of a radioactive isotope if it takes 7 years for 560 grams to decay to 35 grams?
15. The mass in grams of radioactive isotope in a sample is represented by \( M = 14(0.96)^t \) where \( t \) is the time in minutes.

a) State the initial mass of the sample

b) Determine the mass, to the nearest tenth of a gram, remaining after one hour.

c) Determine the time taken, to the nearest minute, for the initial mass to reduce by one-half.

d) Write the equation for \( M \) as an exponential function with base \( \frac{1}{2} \).

e) Use the equation in d) to determine the mass remaining after one hour.

16. A current, \( I_0 \) amperes, falls to \( I \) amperes after \( t \) seconds according to the formula \( I = I_0e^{-kt} \). The value of the constant \( k \) to the nearest whole number if a current of 25 amperes falls to 2.5 amperes in 0.01 seconds is ______.

17. The price of a famous brand name camera lens can be found by the equation \( P = 14(1.1)^c \), where \( c \) is the circumference of the lens in centimetres and \( P \) is the price of the lens in dollars. The diameter, to the nearest tenth of a centimetre, of a camera lens which costs $2500 is ______.
18. The tripling period, to the nearest tenth of an hour, of a bacterial culture which grow from 500 cells to 64 000 cells in 50 hours is _____.

19. Radioactive material decays to 40% of its original mass in 5 years. The half-life of the radioactive material, to the nearest hundredth of a year, is _____.

Answer Key

1. 1 377 114  
2. 8  
3. a) $V_t = V_0(0.75)^t$  
b) 11  
4. 16839  
5. 7 filters  
6. a) 0.077  
b) 21.4%  
7. 49.3 million  
8. a) 140°C  
b) 884°C  
9. 149 grams  
10. a) 48 min.  
b) 5:56 a.m.  
c) 5:44 a.m.  
11. 225 min  
12. 32 min  
13. 4.12 g  
14. 21 months  
15. a) 14 grams  
b) 1.2 grams  
c) 17 minutes  
d) $M = 14 \left(\frac{1}{2}\right)^{\frac{t}{17}}$  
e) 1.2 grams  
16. 230  
17. 17.3  
18. 11.3  
19. 3.78
The Richter scale, named after the American seismologist Charles Richter (1900-85), is used to measure the magnitude of an earthquake. The magnitude of an earthquake is a measure of the amount of energy released. It is determined from the logarithm of the amplitude of waves recorded by seismographs.

The Richter scale is logarithmic—a difference in one unit in magnitude corresponds to a factor of ten difference in intensity. This means that each whole number step represents a ten-fold increase in intensity. Therefore a magnitude 9 earthquake is ten times larger than a magnitude 8 earthquake, one hundred times larger than a magnitude 7 earthquake, and one thousand times larger than a magnitude 6 earthquake.

A magnitude of 1.0 on the Richter Scale is equivalent to a large blast at a construction site using approximately 14 kg of TNT. It is 10 times as intense as the zero reference point.

An earthquake with magnitude 2.0 is \(10^2\) times as intense as the zero reference point, etc.

Any earthquakes having magnitudes in excess of 6.0 are considered dangerous. The largest yet recorded, the Chilean earthquake of 1960, registered 9.4 on the Richter scale. The most powerful recorded in North America, the Alaska quake of 1964, reached 8.4 on the Richter scale.

Class Ex. #1

Complete the following.

a) An earthquake of magnitude 8 is ____ times as intense as an earthquake of magnitude 7.

b) An earthquake of magnitude 7 is ____ times as intense as an earthquake of magnitude 3.

c) An earthquake of magnitude 4 is ____ times as intense as an earthquake of magnitude 6.

d) An earthquake of magnitude 4.8 is ____ times as intense as an earthquake of magnitude 6.8.

e) The 1976 earthquake in Italy was ____ times as intense as the 1983 earthquake in Columbia.

f) The 1933 earthquake in Japan was ____ times as intense as the 1966 earthquake in Turkey.
Comparing Earthquake Intensities

We have seen how to compare the intensity of earthquakes whose magnitude differs by an integer.

But how would we compare the intensities of the 1985 Mexico earthquake (magnitude 8.1) and the 1966 Turkey earthquake (magnitude 6.9)?

The magnitude of an earthquake is given by the formula \( M = \log \left( \frac{I}{I_0} \right) \), where \( I \) is the earthquake intensity and \( I_0 \) is a reference intensity.

To compare the intensities of magnitude \( M_1 \) and \( M_2 \), consider the following:

\[
M_1 = \log \left( \frac{I_1}{I_0} \right) \quad \text{and} \quad M_2 = \log \left( \frac{I_2}{I_0} \right)
\]

\[
M_1 - M_2 = \log \left( \frac{I_1}{I_0} \right) - \log \left( \frac{I_2}{I_0} \right)
\]

\[
= (\log I_1 - \log I_0) - (\log I_2 - \log I_0)
\]

\[
= \log I_1 - \log I_2
\]

Changing to exponential form provides the formula

\[
\frac{I_1}{I_2} = 10^{M_1 - M_2}
\]

Class Ex. #2

How many times more intense was the 1985 Mexico earthquake (magnitude 8.1) then the 1966 Turkey earthquake (magnitude 6.9)? Answer to the nearest whole number.

Class Ex. #3

A major earthquake of magnitude 7.5 is 375 times as intense as a minor earthquake. Find the magnitude, to the nearest tenth, of the minor earthquake.
The loudness of a sound was originally measured in **Bels**, named after Alexander Graham Bell.

The current unit used is the **decibel (dB)**, which is equal to one tenth of a Bel.

The Bel scale, like the Richter scale, is logarithmic - a difference of 1 Bel, or 10 decibels, corresponds to a factor of ten difference in sound intensity.

A leaf rustling (10 decibels or 1 Bel) is 10 times as loud as the threshold of hearing.

A whisper (30 decibels or 3 Bels) is $10^3$, or 1000, times as intense as the threshold of hearing and $10^2$, or 100, times as loud as a leaf rustling.

Complete the following using the chart above.

**a)** A power saw (120 dB) is _____ times as loud as a telephone dial tone (80 dB).

**b)** A jet engine at 20 m (145 dB) is _____ times as loud as the threshold of pain (125 dB).

**c)** A whisper (30 dB) is _____ times as loud as a conversation (60 dB).
Comparing Loudness of Sounds

We have seen how to compare the loudness of sound whose Bels differs by an integer. To compare the loudness of sounds when the Bels are not integers, we suggest the following approach.

The loudness, or intensity, of sound can be measured in dB or Bels. The intensity of two sounds can be compared using the following formulas where \( I \) represents sound intensity, and \( I_0 \) represents a reference sound intensity (threshold of hearing):

For dB
\[
\text{dB} = 10 \log \left( \frac{I}{I_0} \right)
\]

To compare the loudness, or intensity, of two sounds measured in **decibels**, use the formula
\[
\frac{I_1}{I_2} = 10^{\frac{\text{dB}_1 - \text{dB}_2}{10}}
\]

For Bels
\[
\text{Bels} = \log \left( \frac{I}{I_0} \right)
\]

To compare the loudness, or intensity, of two sounds measured in **Bels**, use the formula
\[
\frac{I_1}{I_2} = 10^{B_1 - B_2}
\]

Deriving the above formulas is similar to that in the previous section.

Class Ex. #5
Given that \( \text{dB} = 10 \log \left( \frac{I}{I_0} \right) \), show that the decibel level for the threshold of pain (125 dB), has a sound intensity \( 10^{12.5}I_0 \).

Class Ex. #6
How many times more intense is the sound of a piano playing (67 dB) than a whisper (22 dB)?

Class Ex. #7
Two telephones in a home ring at the same time with a loudness of 80 decibels each. Does this mean that the total loudness is 160 dB? Explain why or why not using sound intensities and the properties of logarithms.
In 1909, Sören Sörenson, a Dutch chemist, introduced the term pH, representing the expression “the power of hydrogen”, to measure the extreme wide range of hydrogen ion concentration in substances.

The pH scale measures the range of hydrogen ion concentration by determining the acidity or the alkalinity of a solution. The scale measures from 0 to 14 with values below 7 representing increasing acidity, and values above 7 representing increasing alkalinity. The value 7 represents the neutral level on the pH scale where the solution is neither acidic nor alkaline.

Similar to the Richter scale, the pH scale is logarithmic—a difference in one unit of pH corresponds to a factor of ten difference in intensity. This means that each whole number step represents a ten-fold increase in intensity.

Therefore, if vinegar has a pH of 3, than it is ten times as acidic as tomato juice (pH of 4), and one hundred times as acidic as normal rain (pH of 5).

On the other hand, household ammonia (pH of 11.5), is ten times as alkaline as milk of magnesia (pH of 10.5) and one thousand times as alkaline as sea water (pH of 8.5).

Complete the following using the approximate pH values given.

a) Tomato juice is ________ times as acidic as pure water.

b) Eggs are ________ times as alkaline as pure water.

c) Milk of magnesia is ________ times as ________ as blood.

d) ________ is 100 times as acidic as normal rain.

e) Eggs are ________ times as alkaline as washing soda.
Comparing Acidity and Alkalinity of Solutions

We have compared the acidity and alkalinity of solutions whose pH differs by an integer.

But how would we compare the acidity and alkalinity of substances whose pH is a non-integer?

To find how much more acidic or alkaline one solution is to another, use $10^{pH_1 - pH_2}$

Class Ex. #9
Pure water has a pH of 7, swimming pool water has a pH of 7.5, and sea water has a pH level of 8.4.

a) How many times as alkaline is sea water than pure water?

b) How many times as alkaline as swimming pool water is sea water?

Formula for pH

The pH of a solution is defined as $pH = -\log (H^+)$, where the $H^+$ is the hydrogen ion concentration (expressed as moles/litre).

Class Ex. #10
A patient gave a urine sample which was found to have a pH of 5.7. What was the hydrogen ion concentration? Answer in scientific notation using one decimal place.

Class Ex. #11
Determine the pH of a solution, to the nearest tenth, if the hydrogen ion concentration is $3.4 \times 10^{-4}$ mol/L.

Complete Assignment Questions #1 - #12
Assignment

1. An earthquake in Japan was measured at 8.9 on the Richter Scale, and an earthquake in Italy was measured at 6.5. How many times more intense, to the nearest whole number, was the earthquake in Japan than the one in Italy?

2. An earthquake with magnitude 7.2 hit a region in North America. The next day a second earthquake with magnitude 6.5 hits the same region. How many times more intense, to the nearest whole number, was the first earthquake than the second one?

3. An earthquake in Peru had a magnitude of 7.7 on the Richter Scale. The following day a second earthquake with one third of the intensity of the first hits the same region. What was its magnitude to the nearest tenth?

4. How many times more intense is the sound of a referee’s whistle (125 dB) than a train whistle at 200 m (90 dB)? Answer to the nearest whole number.

5. How many times louder is a clarinet (95 dB) than a flute (89 dB)? Answer to the nearest whole number.

6. Use the chart to answer the following to the nearest whole number.

   a) Eggs are how many times as alkaline as blood?

   b) Black coffee is how many times as acidic as milk?

<table>
<thead>
<tr>
<th>Solution</th>
<th>pH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black Coffee</td>
<td>5.1</td>
</tr>
<tr>
<td>Milk</td>
<td>6.6</td>
</tr>
<tr>
<td>Pure Water</td>
<td>7</td>
</tr>
<tr>
<td>Blood</td>
<td>7.5</td>
</tr>
<tr>
<td>Eggs</td>
<td>8</td>
</tr>
</tbody>
</table>
7. A river has a pH value of 6.4 upstream from a chemical factory and a pH value of 5.8 downstream of the factory. Compare the acidity levels.

8. The pH of a solution is defined as \( \text{pH} = -\log (\text{H}^+) \), where the H\(^+\) is the hydrogen ion concentration (expressed as mol/L).
   
   a) If a solution has a hydrogen ion concentration of \( 1.21 \times 10^{-2} \text{ mol/L} \), determine the pH value, to the nearest tenth, of the solution.

   b) A vinegar solution has pH of 3.2. Determine its hydrogen ion concentration in scientific notation to one decimal place.

   c) A weaker vinegar solution is \( \frac{1}{4} \) as acidic as the solution in b). Determine its pH value to the nearest tenth.

9. a) The ionization of pure water is given by the equations:

\[
[\text{H}^+][\text{OH}^-] = 1.0 \times 10^{-14} \quad \text{and} \quad [\text{H}^+] = [\text{OH}^-]
\]

If the pH of a solution is defined as \( \text{pH} = -\log (\text{H}^+) \), prove the pH of pure water is 7.0.

b) Determine the pH, to the nearest tenth, of an acidic solution whose [OH\(^-\)] concentration is \( 3.98 \times 10^{-10} \text{ mol/L} \) if \( [\text{H}^+][\text{OH}^-] = 1.0 \times 10^{-14} \).
10. The number of students in a school $t$ years after the school opens can be modelled by the equation $S = S_0 \log_2 (t + 1) + 1$, where $S_0$ is the original number of students in the school.

a) If there were initially 100 students in the school, how many would be expected after 10 years?

b) How many years will it take for the number of students to reach 800 if the original number of students in the school was 200?

11. If two different jets are flying together at an air show, each with a sound level of 120 decibels, then the approximate total decibel level is

A. 12.3 decibels
B. 24.0 decibels
C. 123 decibels
D. 240 decibels

12. A major earthquake of magnitude 8.2 is 110 times as intense as a minor earthquake. The magnitude, to the nearest tenth, of the minor earthquake is _____.

Answer Key

1. 251  
2. 5  
3. 7.2  
4. 3162  
5. 4  
6. a) 3  
   b) 32  
7. Downstream is 4 times as acidic as upstream.  
8. a) 1.9  
   b) $6.3 \times 10^{-4}$  
   c) 3.8  
9. b) 4.6  
10. a) 446  
    b) 7  
11. C  
12. 6.2
Trigonometry means “three angle measurement”. Rotation angles can be measured in degrees or in radians. The radian measure of an angle will be introduced in lesson 3.

**Degree Measure of an Angle**

Angles can be measured in degrees where 360° is one complete rotation.

A *rotation angle* is formed by rotating an initial arm through an angle $\theta^\circ$ about a fixed point (the vertex).

A *positive angle* results from a counter clockwise rotation.

A *negative angle* results from a clockwise rotation.

The angle shown in the diagram is said to be in *standard position*.

Draw the rotation angle in standard position.

![Class Ex. #1](image)

Angles with the same terminal arm are called *coterminal angles*.

Since 150° is the measure of the smallest positive rotation angle coterminal with the angles in class example #1, it is called the *principal angle*.

The principal angle will always have a measure between 0° and 360°.

There are infinitely many angles that are coterminal with a given angle. These angles will be of the form $(P + 360n)^\circ$ where $P$ is the principal angle and $n$ is an integer.
Class Ex. #2

a) Find the angle, \( \theta \), in the domain \(-360° \leq \theta \leq 360°\), which is:

i) coterminal with \(200°\)

ii) coterminal with \(-290°\)

b) Write an expression that represents all angles in the domain \(\theta \in \mathbb{R}\) that are coterminal with the angles given above.

Class Ex. #3

For each of the following angles determine:

i) the quadrant of the terminal arm of the angle

ii) the principal angle

a) 265°

b) -111°

c) 1085°

Pythagorean Theorem

The traditional formula for the Pythagorean Theorem is \(c^2 = a^2 + b^2\). In trigonometry we use \(x, y, \) and \(r\) instead of \(a, b, \) and \(c\).

\[x^2 + y^2 = r^2, \text{ where } r > 0\]
Trigonometric Ratios

Primary Trigonometric Ratios

Complete the following

sine ratio \( \Rightarrow \) \( \sin \theta = \)

hypotenuse

side opposite to \( \theta \)

cosine ratio \( \Rightarrow \) \( \cos \theta = \)

side adjacent to \( \theta \)

tangent ratio \( \Rightarrow \) \( \tan \theta = \)

These ratios are called the Primary Trigonometric Ratios and can be remembered by the acronym SOHCAHTOA.

Reciprocal Trigonometric Ratios - are the reciprocals of the primary trigonometric ratios.

\[
\begin{align*}
\text{cosecant ratio} & \quad \Rightarrow \quad \csc \theta = \frac{1}{\sin \theta} \\
\text{secant ratio} & \quad \Rightarrow \quad \sec \theta = \frac{1}{\cos \theta} \\
\text{cotangent ratio} & \quad \Rightarrow \quad \cot \theta = \frac{1}{\tan \theta}
\end{align*}
\]

- We can remember the reciprocal from each of the primary trig ratios by the fact that each “pair” has only one “co” prefix in it.
- The primary and reciprocal trigonometric ratios can be given in terms of \( x, y \) and \( r \).

Note

You should memorize these formulas.
The point $(15, 8)$ lies on the terminal arm of $\theta$ as shown. Calculate the value of $r$ and hence determine the exact values of the primary and reciprocal trigonometric ratios.

$$
\begin{align*}
\text{Class Ex. #5} \\
\text{y} & \quad \text{x} \\
(15, 8) & \quad r \\
\theta & \quad \theta \\
15 & \quad 8
\end{align*}
$$

Angle $A$ terminates in the first quadrant with $\sin A = \frac{1}{3}$. Sketch a diagram and find the other five trigonometric ratios for angle $A$.

Using SOHCAHTOA to Find Trigonometric Ratios, Angles and Sides

Find the length of $m$ to the nearest tenth.

Use the diagram to determine:

a) $\sin A$

b) the measure of angle $A$ to the nearest degree.

Complete Assignment Questions #1 - #11
Assignment

1. Give the measure of each angle in degrees

2. Draw the following rotation angles in standard position.

3. Determine the angle \( \theta, -360° \leq \theta \leq 360° \), which is coterminal with each of the following angles.

4. Write an expression which represents all angles \( \theta, \quad \theta \in \mathbb{R} \), that are coterminal with the angles given in #3.

5. Which of the following angles are coterminal with 80°?

i) 80°
ii) –100°
iii) –280°
iv) 280°
6. For each of the following angles determine the following:
   (i) the quadrant of the terminal arm of the angle
   (ii) the principal angle
   
   a) $-55^\circ$  
   b) $770^\circ$  
   c) $-490^\circ$  
   d) $451^\circ$

7. For each angle a point on the terminal arm is shown. Calculate the exact values of the primary and reciprocal trigonometric ratios.

\[
\begin{array}{c|c|c}
\text{Ratio} & \text{(a)} & \text{(b)} \\
\hline
\sin A & & \\
\cos A & & \\
\tan A & & \\
\csc A & & \\
\sec A & & \\
\cot A & & \\
\end{array}
\]
8. Solve for the required ratios in each of the following. Express each answer as an exact value with a rational denominator.

a) $A$ is a first quadrant angle. If $\tan A = \frac{3}{4}$, find $\sin A$, $\csc A$, and $\sec A$.

b) $\theta$ is a first quadrant angle. If $\cos \theta = \frac{1}{\sqrt{3}}$, find $\sin \theta$, $\tan \theta$, and $\sec \theta$.

9. Calculate the value of $x$ to two decimal places.

a) 

b) 

![Diagram 1] 

![Diagram 2]
10. Angle $P$ has a terminal arm in the first quadrant. If $\sec P = 2$, the value of $\sin P$ is:

A. $\frac{1}{\sqrt{5}}$  
B. $\frac{2}{\sqrt{5}}$  
C. $\frac{1}{2}$  
D. $\frac{\sqrt{3}}{2}$

11. A tower 23.2 m high casts a shadow 15.3 m long. The angle of elevation of the sun, to the nearest tenth of a degree, is _____ .

**Answer Key**

1. a) $771^\circ$  b) $600^\circ$  c) $-417^\circ$  d) $-958^\circ$

2. a) $135^\circ$  b) $300^\circ$  c) $-120^\circ$  d) $570^\circ$  e) $-570^\circ$

3. a) $50^\circ$  b) $-260^\circ$  c) $225^\circ$  d) $-290^\circ$

4. a) $(50 + 360n)^\circ$  b) $(100 + 360n)^\circ$  c) $(225 + 360n)^\circ$  d) $(70 + 360n)^\circ$

5. i), iii) 6. a) quadrant 4, $305^\circ$  b) quadrant 1, $50^\circ$  c) quadrant 3, $230^\circ$  d) quadrant 2, $91^\circ$

7.

<table>
<thead>
<tr>
<th>Ratio</th>
<th>a)</th>
<th>b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin A$</td>
<td>$\frac{12}{13}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\cos A$</td>
<td>$\frac{5}{13}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\tan A$</td>
<td>$\frac{12}{5}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
</tr>
<tr>
<td>$\csc A$</td>
<td>$\frac{13}{12}$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\sec A$</td>
<td>$\frac{13}{5}$</td>
<td>$\frac{2\sqrt{3}}{3}$</td>
</tr>
<tr>
<td>$\cot A$</td>
<td>$\frac{5}{13}$</td>
<td>$\sqrt{3}$</td>
</tr>
</tbody>
</table>

8. a) $\sin A = \frac{3}{5}$  b) $\sin \theta = \frac{\sqrt{3}}{3}$

$csc A = \frac{5}{3}$  $\tan \theta = \sqrt{2}$

$sec A = \frac{5}{4}$  $sec \theta = \sqrt{3}$

9. a) 16.60  b) 8.07  10. D  11. 56.6
Trigonometry - Functions and Graphs Lesson #2: Reference Angles and the CAST Rule

**Warm-Up #1**

a) Use a calculator to find the value of \( \sin 30^\circ \) and \( \sin 150^\circ \) and compare the answers.

b) Compare the value of \( \cos 40^\circ \) and \( \cos 140^\circ \) (to four decimal places).

- To show why the above answers are the same, we will construct and analyze the graphs of \( y = \sin x \) and \( y = \cos x \) from \( 0^\circ \) to \( 360^\circ \).

- Use the following mode and window settings.

**Warm-Up #2**

*Exploring the Graph of \( \sin x \)*

a) Sketch the graph of \( y = \sin x \), \( 0^\circ \leq x \leq 360^\circ \), on the grid.

b) Use the trace feature to complete the table below to four decimal places where necessary.

Press \( \text{Trace} \), enter the value of \( x \), then press \( \text{Enter} \) to find the value of \( y \).

<table>
<thead>
<tr>
<th>( x ) (angle in degrees)</th>
<th>( y ) (sine ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 30^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 45^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 60^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 90^\circ )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x ) (angle in degrees)</th>
<th>( y ) (sine ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 120^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 135^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 150^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 180^\circ )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x ) (angle in degrees)</th>
<th>( y ) (sine ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 210^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 225^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 240^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 270^\circ )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x ) (angle in degrees)</th>
<th>( y ) (sine ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 300^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 315^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 330^\circ )</td>
<td></td>
</tr>
<tr>
<td>( 360^\circ )</td>
<td></td>
</tr>
</tbody>
</table>

c) Without using a calculator, state two angles (not in the table) which have:

i) the same positive value for the sine ratio.

ii) the same negative value for the sine ratio.

Graphing primary and secondary trigonometric functions will be studied in more detail later on in this unit.
**Reference Angles**

In order to investigate pairs of angles with identical trigonometric ratios, we introduce the concept of a reference angle. A reference angle is the acute angle formed between the terminal arm of the rotation angle and the x-axis.

In each case, sketch the rotation angle and state the reference angle.

- **a)** $120^\circ$
- **b)** $243^\circ$
- **c)** $337^\circ$
- **d)** $70^\circ$
- **e)** $-203^\circ$
- **f)** $537^\circ$

Determine the positive rotation angle, $x$, $0^\circ \leq x < 360^\circ$, given the reference angle and the quadrant.

<table>
<thead>
<tr>
<th>Reference Angle</th>
<th>Quadrant</th>
<th>Sketch</th>
<th>Rotation Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$25^\circ$</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$8^\circ$</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$39^\circ$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>between 3 and 4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Complete Assignment Question #1 - #4
Warm-Up #3  
Determining the Sign of a Trigonometric Ratio

a) In each quadrant draw the rotation angle $\theta$ in standard position.

b) Complete the chart to determine the sign of each ratio.
Use $\sin \theta$ in Quadrant 1 as an example.

```
<table>
<thead>
<tr>
<th>Quadrant 1</th>
<th>Quadrant 2</th>
<th>Quadrant 3</th>
<th>Quadrant 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta =$</td>
<td>$\sin \theta =$</td>
<td>$\sin \theta =$</td>
<td>$\sin \theta =$</td>
</tr>
<tr>
<td>$\cos \theta =$</td>
<td>$\cos \theta =$</td>
<td>$\cos \theta =$</td>
<td>$\cos \theta =$</td>
</tr>
<tr>
<td>$\tan \theta =$</td>
<td>$\tan \theta =$</td>
<td>$\tan \theta =$</td>
<td>$\tan \theta =$</td>
</tr>
</tbody>
</table>
```

Use the following quadrants:

- Quadrant 1: $\sin \theta$, $\cos \theta$, $\tan \theta$ are all positive.
- Quadrant 2: $\sin \theta$, $\tan \theta$ are positive, $\cos \theta$ is negative.
- Quadrant 3: $\sin \theta$ is negative, $\cos \theta$, $\tan \theta$ are positive.
- Quadrant 4: $\sin \theta$, $\cos \theta$ are negative, $\tan \theta$ is positive.

c) Complete the following statements using the results from a).

i) Sine ratios have **positive** values in quadrants _____ and _____.

ii) Cosine ratios have **positive** values in quadrants _____ and _____.

iii) Tangent ratios have **positive** values in quadrants _____ and _____.

iv) Sine ratios have **negative** values in quadrants _____ and _____.

v) Cosine ratios have **negative** values in quadrants _____ and _____.

vi) Tangent ratios have **negative** values in quadrants _____ and _____.

### CAST Rule

The results can be memorized by:

- the CAST rule or
- by remembering to “Add Sugar To Coffee”

The reciprocal trigonometric ratios follow the same framework as their corresponding primary ratio.
The trigonometric ratios of any angle can be written as the same function of a positive acute angle called the reference angle with the sign of the ratio being determined by the CAST rule.

Class Ex. #3

Rewrite as the same trigonometric function of a positive acute angle.

a) \(\sin 140^\circ\)  
b) \(\tan 323^\circ\)  
c) \(\cos 235^\circ\)

d) \(\sin (-105)^\circ\)  
e) \(\sec 358^\circ\)  
f) \(\cot 107^\circ\)

Complete Assignment Questions #5 - #8

Assignment

1. Use a graphing calculator with the settings for Warm-Up #1 to answer the following.

a) Sketch the graph of \(y = \cos x, 0^\circ \leq x \leq 360^\circ\), on the grid.

b) Complete the table to four decimal places where necessary.

<table>
<thead>
<tr>
<th>(x) (angle in degrees)</th>
<th>(y) (cosine ratio)</th>
<th>(x) (angle in degrees)</th>
<th>(y) (cosine ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0(^\circ)</td>
<td></td>
<td>120(^\circ)</td>
<td></td>
</tr>
<tr>
<td>30(^\circ)</td>
<td></td>
<td>135(^\circ)</td>
<td></td>
</tr>
<tr>
<td>45(^\circ)</td>
<td></td>
<td>150(^\circ)</td>
<td></td>
</tr>
<tr>
<td>60(^\circ)</td>
<td></td>
<td>180(^\circ)</td>
<td></td>
</tr>
<tr>
<td>90(^\circ)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(x) (angle in degrees)</th>
<th>(y) (cosine ratio)</th>
<th>(x) (angle in degrees)</th>
<th>(y) (cosine ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>210(^\circ)</td>
<td></td>
<td>300(^\circ)</td>
<td></td>
</tr>
<tr>
<td>225(^\circ)</td>
<td></td>
<td>315(^\circ)</td>
<td></td>
</tr>
<tr>
<td>240(^\circ)</td>
<td></td>
<td>330(^\circ)</td>
<td></td>
</tr>
<tr>
<td>270(^\circ)</td>
<td></td>
<td>360(^\circ)</td>
<td></td>
</tr>
</tbody>
</table>

c) Without using a calculator, state two angles (not in the table) which have:

i) the same positive value for the cosine ratio.

ii) the same negative value for the cosine ratio.
2. Use a graphing calculator in degree mode with window setting \( x: [0, 360, 45] \) and \( y: [-10, 10, 1] \) to answer the following.

   a) Sketch the graph of \( y = \tan x, \; 0^\circ \leq x \leq 360^\circ \), on the grid.

   b) Complete the table to four decimal places where necessary.

<table>
<thead>
<tr>
<th>( x ) (angle in degrees)</th>
<th>( y ) (tan ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0.0000</td>
</tr>
<tr>
<td>30°</td>
<td>0.5774</td>
</tr>
<tr>
<td>45°</td>
<td>1.0000</td>
</tr>
<tr>
<td>75°</td>
<td>3.7321</td>
</tr>
<tr>
<td>85°</td>
<td>80.5149</td>
</tr>
<tr>
<td>89°</td>
<td>1.8562</td>
</tr>
<tr>
<td>89.99°</td>
<td>1.8545</td>
</tr>
<tr>
<td>90°</td>
<td>0.0000</td>
</tr>
<tr>
<td>200°</td>
<td>2.1898</td>
</tr>
<tr>
<td>210°</td>
<td>3.9071</td>
</tr>
<tr>
<td>225°</td>
<td>1.1082</td>
</tr>
<tr>
<td>255°</td>
<td>0.9771</td>
</tr>
<tr>
<td>265°</td>
<td>0.2573</td>
</tr>
<tr>
<td>269°</td>
<td>0.0000</td>
</tr>
<tr>
<td>269.99°</td>
<td>0.0000</td>
</tr>
<tr>
<td>270°</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

   c) State the equations of the asymptotes of the graph of \( y = \tan x, \; 0^\circ \leq x \leq 360^\circ \).

   d) Without using a calculator, state two angles (not in the table) which have:

   i) the same positive value for the tangent ratio.

   ii) the same negative value for the tangent ratio.

3. Find the reference angle for the following rotation angles.

   a) 135°  b) 296°  c) 237°  d) −25°  

   e) −245°  f) 820°  g) 180°  h) −270°  

   i) 0°  j) −90°  k) 270°  l) −360°
4. Complete the following tables given the reference angle and the quadrant.

<table>
<thead>
<tr>
<th>Reference Angle</th>
<th>Quadrant</th>
<th>Sketch</th>
<th>Rotation Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25°</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15°</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36°</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4°</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>89°</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0°</td>
<td>between 2 and 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>90°</td>
<td>between 1 and 2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Complete the following statements.
   a) Secant ratios have positive values in quadrants _____ and _____.
   b) Cosecant ratios have positive values in quadrants _____ and _____.
   c) Cotangent ratios have positive values in quadrants _____ and _____.

6. In which quadrant(s) does the terminal arm lie if:
   a) \( \sin \theta \) is negative?  
   b) \( \sec \theta \) is positive?  
   c) \( \csc \theta \) and \( \tan \theta \) are both negative?  
   d) \( \cot \theta \) is positive and \( \csc \theta \) is negative?
7. Rewrite as the same trigonometric function of a positive acute angle.

a) \( \sin 205^\circ = \)

b) \( \cot 193^\circ = \)

c) \( \sec 107^\circ = \)

d) \( \csc (-380^\circ) = \)

e) \( \cos 451^\circ = \)

f) \( \tan (-30^\circ) = \)

8. Which of the following is \( \tan (-105^\circ) \) expressed as the same trigonometric function of a positive acute angle?

A. \( \tan 255^\circ \)
B. \( \tan 15^\circ \)
C. \( \tan 75^\circ \)
D. \( -\tan 75^\circ \)
Answer Key

1. a) [Diagram]
   c) answers may vary
      i) 70°, 290°
      ii) 130°, 230°

2. a) [Diagram]
   c) $x = 90°, x = 270°$
   d) answers may vary
      i) 70°, 250°
      ii) 130°, 310°

3. a) 45°
   b) 64°
   c) 57°
   d) 25°
   e) 65°
   f) 80°
   g) 0°
   h) 90°
   i) 0°
   j) 90°
   k) 90°
   l) 0°

4. [Table]

5. a) 1 and 4
   b) 1 and 2
   c) 1 and 3

6. a) quadrant 3 and 4
   b) quadrant 1 and 4
   c) quadrant 4
   d) quadrant 3

7. a) $-\sin 25°$
   b) $\cot 13°$
   c) $-\sec 73°$
   d) $-\csc 20°$
   e) $-\cos 89°$
   f) $-\tan 30°$

8. C
The reference angle for any rotation angle is the acute angle between the terminal arm of the rotation angle and the x-axis.

- Memorize the relationship between reference angle and rotation angle in each quadrant.
- The relationship given between reference angle and rotation angle applies when the rotation angle is converted (where necessary) to a principal angle.

The CAST rule may be used to determine the sign of a trigonometric ratio.

- the CAST rule or
- by remembering to “Add Sugar To Coffee”

The reciprocal trigonometric ratios follow the same framework as their corresponding primary ratio.
Solving Equations Involving Sine or Cosine

We can use the concepts of reference angles and signs of the trigonometric ratio to solve equations involving sine or cosine. The following procedure may be used to solve an equation such as \( \sin x = 0.5 \), where \( 0^\circ \leq x \leq 360^\circ \).

**Step 1:** Determine the quadrant(s) the angle will be in by looking at the sign of the ratio.

**Step 2:** Determine the reference angle (always between \( 0^\circ \) and \( 90^\circ \)) and draw a rough sketch in the appropriate quadrant(s). The reference angle is found as follows:

- Use \( \text{2nd} \sin \) or \( \text{2nd} \cos \) of the absolute value of the given quantity.

**Step 3:** Determine the rotation angle(s) using the reference angle and the quadrant(s).

- Always check the given domain to determine which quadrants are valid in the calculation. As an example, sometimes the domain is restricted to \( 0^\circ \leq x \leq 180^\circ \), or \( 90^\circ \leq x \leq 180^\circ \).

- Pressing \( \sin \) or \( \cos \) then the angle \( \text{Enter} \) gives the value of the trigonometric ratio for that angle, whereas pressing \( \text{2nd} \sin \) or \( \text{2nd} \cos \) then the given value \( \text{Enter} \) determines the reference angle.

**Class Ex. #1**

Use the above procedure to solve \( \sin x = 0.5 \), where \( 0^\circ \leq x \leq 360^\circ \)

**Class Ex. #2**

Find the measure of \( \theta \), to the nearest whole number, where \( 0^\circ \leq \theta \leq 360^\circ \).

- a) \( \sin \theta = -0.8090 \)
- b) \( \cos \theta = -0.8090 \)
- c) \( \tan \theta = -2.4586 \)
Find the measure of $A$, to the nearest whole number, where $0^\circ \leq A \leq 360^\circ$.

a) $\cot A = 0.5$

b) $\csc A = -2.86$

---

Find the measure of $\theta$, to the nearest whole number, where $0^\circ \leq \theta \leq 360^\circ$.

a) $\csc \theta$ is undefined

b) $\cos \theta = 0$

c) $\sin^2 \theta = \frac{1}{2}$

---

Complete Assignment Questions #1 - #5
Finding the Exact Values of Trigonometric Ratios

Class Ex. #5
Given that the point \((-3, 2)\) lies on the terminal arm of \(\theta\), find the values of the primary and reciprocal trigonometric ratios.

Class Ex. #6
Angle \(A\) terminates in the third quadrant with \(\sin A = -\frac{2}{3}\). Sketch a diagram and find the other five trigonometric ratios for angle \(A\).

Class Ex. #7
If \(\tan \theta = -\frac{3}{2\sqrt{5}}\) and \(\sec \theta\) is positive, then find the value of \(\sin \theta\).

Complete Assignment Questions #6 - #11
Assignment

1. Find the measure of $\theta$, to the nearest degree, where $0^\circ \leq \theta \leq 360^\circ$.
   a) $\sin \theta = 0.6485$
   b) $\cos \theta = -0.8219$
   c) $\tan \theta = 0.4668$
   d) $\csc \theta = 1.0138$

2. Find the measure of $A$, to the nearest degree, where $0^\circ \leq A \leq 360^\circ$.
   a) $\sec A = 1.2364$
   b) $\cot A = -0.4458$
   c) $\sin A = -1$
   d) $\cot A$ is undefined

3. Find the measure of $\theta$, to the nearest degree, where $0^\circ \leq \theta \leq 360^\circ$.
   a) $\tan^2 \theta = 3$
   b) $\sec^2 \theta = \frac{4}{3}$
4. Find the value of the following trigonometric ratios to 4 decimal places.
   a) $\tan 30^\circ$ b) $\sin (-180^\circ)$ c) $\cot 78^\circ$ d) $\sec 271^\circ$ e) $\csc (-205^\circ)$

5. Find the value of $x$. Note that $x$ represents either an angle, where $0^\circ \leq x \leq 360^\circ$, or the value of a trigonometric ratio.
   a) $\csc 150^\circ = x$ b) $\cot x = -0.5638$ c) $x = \sec 348^\circ$
   d) $1.0736 = \tan x$ e) $\sin (-184^\circ) = x$ f) $\sin x = 0$

6. Solve for the required ratios in each of the following. Express each answer as an exact value with a rational denominator.
   a) $\theta$ is a second quadrant angle. If $\tan \theta = -\frac{\sqrt{3}}{5}$, find $\cos \theta$, $\csc \theta$, and $\cot \theta$. 
b) \( \theta \) is a fourth quadrant angle. If \( \tan \theta = -\frac{\sqrt{3}}{5} \), find \( \cos \theta \), \( \csc \theta \), and \( \cot \theta \).

c) If \( \cos A = -\frac{7}{25} \), and \( 180^\circ \leq A \leq 270^\circ \), find the values of \( \sin A \), \( \tan A \), and \( \sec A \).

7. If \( \sin X = -\frac{1}{4} \) and \( \cot X \) is positive, express \( \cos X \) as an exact value.
8. The solution of the equation \( \sec x = -10.366 \) in the interval \( 0^\circ \leq x \leq 360^\circ \) is
   A. \( 84^\circ, 276^\circ \)
   B. \( 96^\circ, 264^\circ \)
   C. \( 96^\circ, 276^\circ \)
   D. \( 264^\circ, 276^\circ \)

9. If \( \csc \theta = -2 \), \( 270^\circ \leq \theta \leq 360^\circ \) the value of \( \cos \theta \) is
   A. \( -\frac{\sqrt{3}}{2} \)
   B. \( \frac{\sqrt{3}}{2} \)
   C. \( \frac{1}{2} \)
   D. \( \frac{\sqrt{3}}{2} \)

10. Angle \( P \) has a terminal arm in the third quadrant.
    If \( \cot P = \sqrt{3} \), the value of \( \sin P - \cos P \) is:
    A. \( \frac{1 - \sqrt{3}}{2} \)
    B. \( \frac{\sqrt{3} - 1}{2} \)
    C. \( \frac{1 + \sqrt{3}}{2} \)
    D. \( \frac{-1 - \sqrt{3}}{2} \)

11. For \( 360^\circ < \theta < 540^\circ \), the solution, to the nearest degree, of the equation
    \( \cos \theta = -\frac{1}{3} \) is ____.
Answer Key

1. a) 40°, 140°  
   b) 145°, 215°  
   c) 25°, 205°  
   d) 81°, 99°

2. a) 36°, 324°  
   b) 114°, 294°  
   c) 270°  
   d) 0°, 180°, 360°

3. a) 60°, 120°, 240°, 300°  
   b) 30°, 150°, 210°, 330°

4. a) 0.5774  
   b) 0.0000  
   c) 0.2126  
   d) 57.2987  
   e) 2.3662

5. a) 2  
   b) 119°, 299°  
   c) 1.0223  
   d) 47°, 227°  
   e) 0.0698  
   f) 0°, 180°, 360°

6. a) \( \cos \theta = \frac{5\sqrt{7}}{14} \),  
    \( \csc \theta = \frac{2\sqrt{21}}{3} \),  
    \( \cot \theta = -\frac{5\sqrt{3}}{3} \)

   b) \( \cos \theta = \frac{5\sqrt{7}}{14} \),  
    \( \csc \theta = \frac{2\sqrt{21}}{3} \),  
    \( \cot \theta = \frac{5\sqrt{3}}{3} \)

   c) \( \sin A = \frac{-24}{25} \),  
    \( \tan A = \frac{24}{7} \),  
    \( \sec A = -\frac{25}{7} \)

7. \( -\frac{\sqrt{15}}{4} \)

8. B  
9. D  
10. B  
11. 469
Warm-Up #1

i) What is the formula for the circumference of a circle of radius \( r \)?

ii) Find the circumference of the following circle. Leave your answer as an exact value in terms of \( \pi \).

\[
\text{circumference} = 2\pi r
\]

Warm-Up #2

In all previous work with angular measure we have used degree measure. One degree is defined as \( \frac{1}{360} \) of a revolution.

In order to simplify some of the calculations involved in trigonometry and calculus, mathematicians use an alternative angular measure - radian measure.

The Radian Measure of an Angle

The radian measure of an angle is a ratio that compares the length of an arc of a circle to the radius of the circle, i.e.

\[
\text{measure of an angle in radians} = \frac{\text{length of arc subtending the angle}}{\text{length of radius}}
\]

• The radian measure of \( \angle AOB \) is given by the ratio \( \frac{\text{arc } AB}{\text{radius } OA} \) (see diagram 1)

• One radian is the measure of the angle at the centre of a circle subtended by an arc equal in length to the radius of the circle (see diagram 2)

\[
\angle POQ = 1 \text{ radian}
\]

• Use diagram 1 and the definition to estimate the radian measure of \( \angle AOB \).

• Use diagram 2 to estimate the degree measure of \( \angle POQ \).
Converting Between Degrees and Radians

Since an angle can be measured in degrees or radians, it is important in to be able to convert from one measure to the other.

**Warm-Up #3**

Consider a circle with a radius of $r$ units. Complete the following:

a) i) one complete rotation in degrees is _________.

   ii) the arc length for one complete rotation is _______ which is the __________________ of the circle.

   iii) the radian measure of an angle of $360^\circ$ is _______.

b) i) one-half rotation in degrees is _________.

   ii) the arc length for one-half rotation is _______.

   iii) the radian measure of an angle of $180^\circ$ is _______.

**Note**

- In mathematics, the symbol “°” following a number means the unit of angular measure is degrees.
- If there is no unit after the number, or there is the abbreviation “rad”, or the word radians, then the unit is radians.
- For example, if you wish to write the sine ratio for a right angle, you must write \( \sin 90^\circ \), and NOT \( \sin 90 \).

---

**Class Ex. #1**

a) Complete the chart:

<table>
<thead>
<tr>
<th>Degrees</th>
<th>360°</th>
<th>180°</th>
<th>90°</th>
<th>60°</th>
<th>45°</th>
<th>30°</th>
<th>1°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) We note that the rule to convert from **degrees to radians** is to multiply the angle in degrees by

---

**Class Ex. #2**

Convert from degrees to radians (give your answer as an exact value in terms of $\pi$)

a) $270^\circ$  

b) $315^\circ$

---

**Class Ex. #3**

Convert the following from degrees to radians (to the nearest tenth)

a) $70^\circ$  

b) $205^\circ$
a) Complete the chart:

<table>
<thead>
<tr>
<th>Radians</th>
<th>$2\pi$</th>
<th>$\pi$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{180}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degrees</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) We note that the rule to convert from **radians to degrees** is to multiply the angle in degrees by

Convert the following from radians to degrees.

a) $\frac{\pi}{4}$

b) $\frac{-7\pi}{3}$

Convert the following radians to degrees (to the nearest tenth)

a) 1.57 radians

b) 3.2

Complete Assignment Questions #1 - #5

a) Use a calculator in **degree mode** to find the value (to 4 decimal places where necessary).

i) $\sin 45^\circ$

ii) $\sec 135^\circ$

b) Use a calculator in **radian mode** to find the value (to 4 decimal places where necessary).

i) $\sin \frac{\pi}{4}$

ii) $\sec \frac{3\pi}{4}$

c) Find the value (to 4 decimal places) of:

i) $\sin \frac{5\pi}{3}$

ii) $\cot 135^\circ$

iii) $\tan \left(\frac{5\pi}{6}\right)$

iv) $\cos 45^\circ$
In each of the following:

i) Draw the angle $\theta$ in standard position

ii) State the principal angle

iii) Find one positive and one negative coterminal angle for the angle $\theta$

\[
a) \quad \theta = \frac{3\pi}{4} \\
b) \quad \theta = -\frac{\pi}{3}
\]

Find the reference angle for the following rotation angles.

\[
a) \quad \frac{5\pi}{6} \\
b) \quad -\frac{5\pi}{4} \\
c) \quad \frac{11\pi}{3}
\]
Assignment

1. Convert from degrees to radians. Express your answer as an exact value in terms of $\pi$.
   a) $30^\circ$  
   b) $45^\circ$  
   c) $60^\circ$  
   d) $135^\circ$  
   e) $240^\circ$  
   f) $150^\circ$  
   g) $90^\circ$  
   h) $270^\circ$  
   i) $225^\circ$  
   j) $420^\circ$

2. Convert from radians to degrees.
   a) $\frac{\pi}{2}$  
   b) $\frac{\pi}{4}$  
   c) $-\frac{2\pi}{3}$  
   d) $\frac{\pi}{6}$  
   e) $\frac{3\pi}{4}$  
   f) $-\frac{3\pi}{2}$  
   g) $\frac{7\pi}{4}$  
   h) $-\frac{5\pi}{6}$

3. Convert from degrees to radians. Give the answers to 1 decimal place.
   a) $50^\circ$  
   b) $205^\circ$  
   c) $57.3^\circ$  
   d) $250^\circ$  
   e) $\left(\frac{120}{\pi}\right)^\circ$

4. Convert from radians to degrees. Give the answers to the nearest tenth.
   a) $0.5$ radians  
   b) $3.1$ rad  
   c) $0.4$  
   d) $1.8\pi$ radians

5. Complete the chart:

<table>
<thead>
<tr>
<th>Angle in Degrees</th>
<th>0º</th>
<th>30º</th>
<th>45º</th>
<th>60º</th>
<th>90º</th>
<th>120º</th>
<th>135º</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle in Radians</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angle in Degrees</td>
<td>150º</td>
<td>180º</td>
<td>210º</td>
<td>225º</td>
<td>240º</td>
<td>270º</td>
<td></td>
</tr>
<tr>
<td>Angle in Radians</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angle in Degrees</td>
<td>300º</td>
<td>315º</td>
<td>330º</td>
<td>360º</td>
<td>540º</td>
<td>720º</td>
<td></td>
</tr>
<tr>
<td>Angle in Radians</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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6. Find the value (to 4 decimal places where necessary) of
   a) $\tan \frac{\pi}{4}$  
   b) $\sin 300^\circ$  
   c) $\cos \frac{3\pi}{4}$  
   d) $\sin \left( \frac{2\pi}{3} \right)$

   e) $\cot 30^\circ$  
   f) $\cos \frac{5\pi}{2}$  
   g) $\csc 60$  
   h) $\sec \left( -\frac{7\pi}{6} \right)$

7. In each of the following :
   i) Draw the angle $\theta$ in standard position
   ii) State the principal angle
   iii) Find one positive and one negative coterminal angle for the angle $\theta$

   a) $\theta = \frac{5\pi}{4}$  
   b) $\theta = \frac{11\pi}{6}$  
   c) $\theta = -\frac{2\pi}{3}$  
   d) $\theta = \frac{14\pi}{3}$

8. Find the reference angle for the following rotation angles.

   a) $\frac{7\pi}{6}$  
   b) $\frac{3\pi}{4}$  
   c) $\frac{11\pi}{6}$  
   d) $-\frac{\pi}{6}$

   e) $-\frac{11\pi}{6}$  
   f) $-\frac{5\pi}{3}$  
   g) $5\pi$  
   h) $\frac{3\pi}{2}$
9. Determine the rotation angle given the reference angle and the quadrant

<table>
<thead>
<tr>
<th>Reference Angle</th>
<th>Quadrant</th>
<th>Rotation Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{8}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{12}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>between 3 and 4</td>
<td></td>
</tr>
</tbody>
</table>

10. An angle with radian measure 2.36 has degree measure of

- A. 424.80
- B. 135.22
- C. 67.61
- D. 0.04

11. Correct to the nearest tenth of a degree, $\frac{3\pi}{8}$ rad is equal to _____ °.
Answer Key

1. a) \(\frac{\pi}{6}\)  b) \(\frac{\pi}{4}\)  c) \(\frac{\pi}{3}\)  d) \(\frac{3\pi}{4}\)  e) \(\frac{4\pi}{3}\)  f) \(\frac{5\pi}{6}\)
   g) \(\frac{\pi}{2}\)  h) \(\frac{3\pi}{2}\)  i) \(\frac{5\pi}{4}\)  j) \(\frac{7\pi}{3}\)

2. a) 90°  b) 45°  c) –120°  d) 30°  e) 135°  f) –270°  g) 315°  h) –150°

3. a) 0.9  b) 3.6  c) 1.0  d) 4.4  e) 0.7

4. a) 28.6°  b) 177.6°  c) 22.9°  d) 324.0°

5.

<table>
<thead>
<tr>
<th>Angle in Degrees</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>135°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle in Radians</td>
<td>0</td>
<td>(\frac{\pi}{6})</td>
<td>(\frac{\pi}{4})</td>
<td>(\frac{\pi}{3})</td>
<td>(\frac{\pi}{2})</td>
<td>(\frac{2\pi}{3})</td>
<td>(\frac{3\pi}{4})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Angle in Degrees</th>
<th>150°</th>
<th>180°</th>
<th>210°</th>
<th>225°</th>
<th>240°</th>
<th>270°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle in Radians</td>
<td>(\frac{5\pi}{6})</td>
<td>(\pi)</td>
<td>(\frac{7\pi}{6})</td>
<td>(\frac{5\pi}{4})</td>
<td>(\frac{4\pi}{3})</td>
<td>(\frac{3\pi}{2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Angle in Degrees</th>
<th>300°</th>
<th>315°</th>
<th>330°</th>
<th>360°</th>
<th>540°</th>
<th>720°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle in Radians</td>
<td>(\frac{5\pi}{3})</td>
<td>(\frac{7\pi}{4})</td>
<td>(\frac{11\pi}{6})</td>
<td>2(\pi)</td>
<td>3(\pi)</td>
<td>4(\pi)</td>
</tr>
</tbody>
</table>

6. a) 1  b) –0.8660  c) –0.7071  d) –0.8660  e) 1.7321  f) 0  g) –3.2807  h) –1.1547

7.

(i) \(\frac{5\pi}{4}\)  (ii) \(\frac{11\pi}{6}\)  (iii) \(\frac{4\pi}{3}\)  (iv) \(\frac{2\pi}{3}\)

(iii) \(\frac{13\pi}{4}\), \(\frac{3\pi}{4}\), \(\frac{23\pi}{6}\), \(\frac{5\pi}{6}\), \(\frac{10\pi}{3}\), \(\frac{8\pi}{3}\), \(\frac{2\pi}{3}\), \(\frac{4\pi}{3}\)  Answers may vary

8. a) \(\frac{\pi}{6}\)  b) \(\frac{\pi}{4}\)  c) \(\frac{\pi}{6}\)  d) \(\frac{\pi}{6}\)  e) \(\frac{\pi}{6}\)  f) \(\frac{\pi}{3}\)  g) 0  h) \(\frac{\pi}{2}\)

9. \(\frac{4\pi}{3}\) \(\frac{\pi}{8}\) \(\frac{11\pi}{6}\) \(\frac{11\pi}{12}\) \(\frac{3\pi}{2}\)  10. B  11. 67.5
Trigonometry - Functions and Graphs Lesson 5: Applications of Radian Measure

Solving Simple Trigonometric Equations in Radian Measure

Class Ex. #1
Find each value of $\theta$ for $0 \leq \theta \leq 2\pi$.

a) $\sin \theta = \frac{1}{2}$

b) $\sec \theta = -\sqrt{2}$

c) $\tan^2 \theta = 3$

Class Ex. #2
Solve the following equations. Give your answer to the nearest hundredth.

a) $\sin x = 0.425$, $0 \leq x \leq 2\pi$

b) $\tan x = -\frac{4}{5}$, $0 \leq x \leq \pi$

Class Ex. #3
Find the value of $\csc x$ if $\tan x = \frac{1}{\sqrt{3}}$ and $\pi \leq x \leq \frac{3\pi}{2}$.

Complete Assignment Questions #1 - #5
Arc Length

In the last lesson we defined the radian measure of an angle as

\[
\text{measure of an angle in radians} = \frac{\text{length of arc subtending the angle}}{\text{length of radius}} \quad \text{i.e.} \quad \theta = \frac{a}{r}
\]

Use the formula below to solve problems involving arc length, radius and central angle.

\[ a = r\theta \]

where, \( \theta \) = the measure of the angle in **radians**

**NOT on the formula sheet**

\( a = \) length of the arc around the angle

\( r = \) length of radius

---

**Class Ex. #4**

A pendulum 30 cm long swings through an arc of 45 cm. Through what angle does the pendulum swing. Answer in degrees and in radians to the nearest tenth.

---

**Class Ex. #5**

A circle of radius 3 cm contains a central angle of 2.4 radians. Calculate the length of the arc subtended by the central angle, to the nearest tenth of a centimetre.

---

**Class Ex. #6**

Calculate the arc length (to the nearest tenth of a metre) of a sector of a circle with diameter 9.2 m if the sector angle is 150°.

---

**Class Ex. #7**

A circle with centre \( C \) and minor arc \( AB \) measuring 15.2 cm is shown. If \( \angle ABC = \angle BAC = \frac{\pi}{6} \) radians, find the length of the radius of the circle to the nearest tenth of a centimetre.

---

**Complete Assignment Questions #6 - #16**
Assignment

1. Find each value of \( \theta \) for \( 0 < \theta < 2\pi \)

   a) \( \sin \theta = \frac{1}{2} \)  
   b) \( \cos \theta = -\frac{1}{\sqrt{2}} \)  
   c) \( \tan \theta = -1 \)

   d) \( 2 \sin \theta = \sqrt{2} \)  
   e) \( \cot \theta = -\sqrt{3} \)  
   f) \( 2 \cos \theta + 1 = 0 \)

   g) \( \csc \theta = 1 \)  
   h) \( \sec \theta - 2 = 0 \)  
   i) \( \sqrt{3} \cot \theta = 1 \)
2. Solve the following equations. Give your answer to the nearest hundredth of a radian.

   a) \tan x = 0.5371, \; 0 \leq x \leq 2\pi  
   b) \cos x = -\frac{4}{5}, \; \pi \leq x \leq 2\pi  

   c) \csc x = 6, \; 0 \leq x \leq \pi  
   d) \cot x = -1.5, \; 0 \leq x \leq 2\pi  

3. Find the values of each angle \( \theta \) if \( 0 \leq \theta \leq 2\pi \).

   a) \sin^2 \theta = \frac{1}{2}  
   b) \tan^2 \theta = \frac{1}{3}  

   c) \sec^2 \theta = \frac{4}{3}  
   d) \csc^3 \theta = -8
4. If \( \tan x = -\sqrt{3} \) and \( \frac{\pi}{2} < x < \pi \), then find the value of \( \csc x \).

5. If \( \sec \theta = -\frac{2}{\sqrt{3}} \) and \( \pi \leq \theta \leq \frac{3\pi}{2} \), what is the value of \( \cot \theta \)?

6. A circle has radius 8 cm. Determine (in radians) the measure of the central angle subtended by an arc of length 5.6 cm.

7. Calculate the arc length (to the nearest tenth of a metre) of a sector of a circle with radius 8.4 m if the sector angle is 80°.

8. In the diagram the circle with centre \( C \) has radius 6 cm. Find the length of arc \( ADB \), to the nearest tenth of a centimetre, if \( \angle CAB = \frac{\pi}{8} \).
9. An arc of a circle is 1.5 cm long and subtends an angle of 3 radians at the centre of the circle. Calculate the radius of the circle.

10. Two sectors of the same circle have sector angles of 30° and 105° respectively. The arc length of the smaller sector is 12 cm.

a) What is the radius of the circle?

b) What is the arc length of the larger sector?

11. A pendulum swings through an angle of 45°. Find the length of the pendulum (to the nearest cm) if the end of the pendulum swings through an arc of length 32 cm.

12. If \(\tan X = -2\) and \(\frac{3\pi}{2} \leq X \leq 2\pi\), the value of \(\csc X\) is

A. \(-\sqrt{5}\)

B. \(\sqrt{5}\)

C. \(-\frac{\sqrt{5}}{2}\)

D. \(\frac{\sqrt{5}}{2}\)
13. An arc of a circle subtends a central angle $x^\circ$. If the length of the arc is 1.2 cm and the diameter of the circle is 4 cm, then the value of $x$ to the nearest whole number is

A. 17
B. 34
C. 54
D. 108

14. An arc $DE$ of a circle, center $O$, is $\frac{1}{6}$ of the circumference. The size of $\angle DOE$, to the nearest one hundredth of a radian, is _____.

15. A person on a Ferris wheel moves a distance of 5 metres from position $P$ to position $Q$. If the diameter of the wheel is 18 metres, the measure of the central angle, to the nearest tenth of a degree, is _____.

16. A satellite makes one complete revolution of the earth in 90 min. Assume that the orbit is circular and that the satellite is situated 280 km above the equator. If the radius of the earth at the equator is 6400 km, then the speed of the satellite, in kilometres per second, to the nearest one hundredth, is _____.
264  Trigonometry - Functions and Graphs Lesson #5:  Applications of Radian Measure

Answer Key

1.  a) \( \frac{\pi}{6} , \frac{5\pi}{6} \)  
   b) \( \frac{3\pi}{4} , \frac{5\pi}{4} \)  
   c) \( \frac{3\pi}{4} , \frac{7\pi}{4} \)  
   d) \( \frac{\pi}{4} , \frac{3\pi}{4} \)  
   e) \( \frac{5\pi}{6} , \frac{11\pi}{6} \)  
   f) \( \frac{2\pi}{3} , \frac{4\pi}{3} \)  
   g) \( \frac{\pi}{2} \)  
   h) \( \frac{\pi}{3} , \frac{5\pi}{3} \)  
   i) \( \frac{\pi}{3} , \frac{4\pi}{3} \)  

2.  a) 0.49, 3.63  
   b) 3.79  
   c) 0.17, 2.97  
   d) 2.55, 5.70

3.  a) \( \frac{\pi}{4} , \frac{3\pi}{4} , \frac{5\pi}{4} , \frac{7\pi}{4} \)  
   b) \( \frac{\pi}{6} , \frac{5\pi}{6} , \frac{5\pi}{6} , \frac{7\pi}{6} , \frac{11\pi}{6} \)  
   c) \( \frac{\pi}{6} , \frac{5\pi}{6} , \frac{7\pi}{6} , \frac{11\pi}{6} \)  
   d) \( \frac{7\pi}{6} , \frac{11\pi}{6} \)  

4. \( \frac{2\sqrt{3}}{3} \)  

5. \( \sqrt{3} \)  

6. 0.7 rad  

7. 11.7 m

8. 14.1

9. 0.5 cm  
10.  a) 22.9 cm  
    b) 42 cm  
11. 41 cm

12. C  
13. B  
14. 1.05  
15. 31.8°  
16. 7.77
Introduction

Our objective is to determine exact values for the trigonometric ratios of angles whose measure is a multiple of $30^\circ \left( \frac{\pi}{6} \text{ rad} \right)$ or a multiple of $45^\circ \left( \frac{\pi}{4} \text{ rad} \right)$.

Warm-up

a) Diagram 1 shows an angle of $45^\circ$ in standard position. An isosceles triangle is drawn whose equal sides are 1 unit.
   i) Determine the length of the hypotenuse.
   ii) Use SOHCAHTOA or the $x, y, r$ formulas to complete:
   \[
   \sin 45^\circ = \cos 45^\circ = \tan 45^\circ =
   \]

b) Diagram 2 shows an angle of $60^\circ$ in standard position. An equilateral triangle is drawn whose equal sides are 2 units and a vertical altitude is drawn which divides the equilateral triangle into two congruent triangles.
   i) Determine the length of the altitude.
   ii) Complete:
   \[
   \sin 60^\circ = \cos 60^\circ = \tan 60^\circ =
   \]

c) Diagram 3 shows an angle of $30^\circ$ in standard position. An equilateral triangles is drawn whose equal sides are 2 units and a horizontal altitude is drawn which divides the equilateral triangle into two congruent triangles.
   i) Complete:
   \[
   \sin 30^\circ = \cos 30^\circ = \tan 30^\circ =
   \]
**Special Triangles**

The following triangles developed on the previous page occur often in trigonometry

![Diagram of triangles]

However if we consider similar triangles to the above, all with hypotenuse length of one unit, we would get the following triangles.

![Diagram of similar triangles]

The triangles above are similar to the ones in the Warm-Up and illustrate the trigonometric ratios as exact values for angles of 30°, 45° and 60°.

In each diagram the horizontal distance is \( x \), the vertical distance is \( y \) and the hypotenuse is \( r = 1 \).

**Finding Exact Primary Trigonometric Values for 0° and 90°**

**a)** Consider a rotation angle of 0°.
In this case \( x = 1, y = 0 \) and \( r = 1 \).

<table>
<thead>
<tr>
<th>( \sin 0° )</th>
<th>( \sin 90° )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos 0° )</td>
<td>( \cos 90° )</td>
</tr>
<tr>
<td>( \tan 0° )</td>
<td>( \tan 90° )</td>
</tr>
</tbody>
</table>

**b)** Consider a rotation angle of 90°.
In this case \( x = 0, y = 1 \) and \( r = 1 \).

**c)** Explain why \( \tan 90° \) is undefined.
Memorizing The Angles and Ratios of Special Triangles

Although there are many ways to memorize the angles and ratios of special triangles, we will discuss two
- by chart
- by the unit circle

Using a Chart for Special Triangles

We can summarize the exact values of trig ratios of 0° (0 rad), 30° \( \frac{\pi}{6} \) rad, 45° \( \frac{\pi}{4} \) rad, 60° \( \frac{\pi}{3} \) rad, and 90° \( \frac{\pi}{2} \) rad in the following chart.

<table>
<thead>
<tr>
<th>x in degrees</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>x in radians</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>sin x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cos x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tan x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This chart should be memorized.

Determining Exact Values of Trigonometric Ratios Using the Chart

We can use the previous table together with the concept of reference angles and the CAST rule to determine the exact values of the trigonometric ratios of certain angles in quadrants 2, 3, and 4.

Class Ex. #1

Complete the solution below to find the exact value of:

a) \( \sin 210° \)

Solution
A rotation angle of 210° has a reference angle of 30°.
In quadrant 3 the sine ratio is negative.
\[ \sin 210° = -\sin 30° = _____ \]

b) \( \cos \frac{5\pi}{3} \)

Solution
A rotation angle of \( \frac{5\pi}{3} \) has a reference angle of ___.
In quadrant ___ the cosine ratio is _____.
\[ \cos \frac{5\pi}{3} = _____ = _____ \]

c) \( \tan (-135°) \)

Complete Assignment Questions #1 - #3

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Using the Unit Circle for Special Triangles

An alternative method for determining exact values for trigonometric ratios of angles greater than 90° is to use the unit circle.

Consider a rotating arm of length 1 unit on the triangles developed from the Warm-Up

In each diagram the;
- horizontal distance is \( x \),
- vertical distance is \( y \),
- hypotenuse is \( r = 1 \).

These triangles can be placed in a circle with a radius of one unit.

The circle above, with a radius of one unit, is called the unit circle and is important you understand how it works.

Recall the formulas \( \sin \theta = \frac{y}{r} \), \( \cos \theta = \frac{x}{r} \), \( \tan \theta = \frac{y}{x} \), and \( \cot \theta = \frac{x}{y} \)

- In the unit circle, where \( r = 1 \), we have:
  \[ \sin \theta = _____ \quad \text{and} \quad \cos \theta = _____ \]

- Every point on the unit circle has coordinates \((x, y)\) which can be written as \((\cos \theta, \sin \theta)\)

  
  \[ \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{and} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \]

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Use the unit circle to find the exact value of all the trigonometric ratios for a rotation angle of 240°. Give each answer with a rational denominator.

\[
\begin{align*}
\sin 240° &= \\
\cos 240° &= \\
\tan 240° &= \\
csc 240° &= \\
sec 240° &= \\
cot 240° &= 
\end{align*}
\]

Use the unit circle to find the exact value of

\(a) \ \cos \frac{3\pi}{4}\) \hspace{1cm} \(b) \ \tan 600°\) \hspace{1cm} \(c) \ \csc 3\pi\)

**Note**
We now have two methods for determining exact values of trigonometric ratios of angles greater than 90°. Use either method.

Use the chart or unit circle to find the exact value of:

\(a) \ \sec 225°\) \hspace{1cm} \(b) \ \cot \frac{5\pi}{3}\) \hspace{1cm} \(c) \ \sin^2 \frac{3\pi}{4} + \cos^2 \frac{3\pi}{4}\)

\(A\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\) and \(B\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\) are two points on the unit circle. If an object rotates counterclockwise from point \(A\) to point \(B\) through what angle has it rotated? Answer in degrees and in radians.

Find the exact value of:

\(a) \ \log_2 \left(\cos \frac{7\pi}{4}\right)\) \hspace{1cm} \(b) \ \log_4 (\csc 510°)\)
Using Exact Values to Solve Simple Trigonometric Equations

Find the exact value(s) of $\theta$ where

a) $\cot \theta = -\sqrt{3}$, $0^\circ \leq \theta \leq 360^\circ$

b) $\sec \theta = -2$, $0 \leq \theta \leq 2\pi$.

Complete Assignment Questions #4 - #15

Assignment

1. Find the exact value of:

   a) $\cos 120^\circ$
   b) $\tan 300^\circ$
   c) $\sin 135^\circ$

   d) $\sin (-30)^\circ$
   e) $\cos^2 225^\circ$
   f) $\tan 480^\circ$

2. Find the exact value of:

   a) $\sin \frac{5\pi}{3}$
   b) $\tan \frac{7\pi}{6}$
   c) $\cos \left( -\frac{2\pi}{3} \right)$

   d) $\sin \left( -\frac{\pi}{6} \right)$
   e) $\tan^2 \frac{2\pi}{3}$
   f) $\cos \left( -\frac{5\pi}{3} \right)$
3. Find the exact value of:
   a) \( \sec 300^\circ \)  
   b) \( \cot \frac{5\pi}{6} \)  
   c) \( \csc \left( \frac{5\pi}{3} \right) \)  
   d) \( \cot 930^\circ \)  
   e) \( \sec \frac{3\pi}{2} \)  
   f) \( \csc 5\pi \)

4. Find exact values for each using the unit circle.
   a) \( \cos 60^\circ \)  
   b) \( \sin 90^\circ \)  
   c) \( \csc \frac{\pi}{6} \)  
   d) \( \cot \frac{7\pi}{4} \)  
   e) \( \sec 150^\circ \)  
   f) \( \tan \frac{5\pi}{3} \)  
   g) \( \csc(-330^\circ) \)  
   h) \( \sec(-135^\circ) \)  
   i) \( \cot 540^\circ \)  
   j) \( \tan \left( -\frac{\pi}{3} \right) \)

5. Find the coordinates of the point on the unit circle that correspond to each rotation. Give the coordinates correct to two decimal places where necessary.
   a) \(-270^\circ \)  
   b) \(103^\circ \)  
   c) \(-16^\circ \)

6. The point \( T(-0.8829, 0.4695) \) lies on the unit circle. Determine the value of \( \theta \) where \( \theta \) is the angle made by the positive \( x \)-axis and the line passing through \( T \).

7. \( P\left( -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \) and \( Q\left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \) are two points on the unit circle. If an object rotates counterclockwise from point \( P \) to point \( Q \) through what angle has it rotated? Answer in degrees and in radians.
8. Determine the length of the arc on the unit circle which has initial point 
   \( A(-1, 0) \) and terminal point \( B\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \).

9. Find the exact value of:
   \( \text{a) } \log_3 \left( \cot \frac{4\pi}{3} \right) \quad \text{b) } \sum_{n=4}^{8} \sin \frac{n\pi}{6} \)

10. Find the exact value(s) of \( \theta \) where \( 0^\circ \leq \theta \leq 360^\circ \).
    \( \text{a) } \sin \theta = 1 \quad \text{b) } \cos \theta = -1 \quad \text{c) } \cot^2 \theta = 1 \quad \text{e) } \sec \theta = \frac{2}{\sqrt{2}} \)

11. Find the measure of \( \theta \) where \( 0 \leq \theta \leq 2\pi \).
    \( \text{a) } \sin \theta = -\frac{\sqrt{3}}{2} \quad \text{b) } \tan \theta = 0 \quad \text{c) } \cos \theta = -\frac{1}{2} \)
    \( \text{d) } \csc \theta = \sqrt{2} \quad \text{e) } \cot \theta = -1 \quad \text{f) } \cot \theta \text{ is undefined} \)
12. If \( \cos \theta = \frac{1}{\sqrt{2}} \), and \( \theta \) is not a first quadrant angle, find the exact values of \( \cot \theta \) and \( \csc \theta \).

13. Correct to the nearest tenth of a radian, the smallest positive root of the equation \( \tan x + \sqrt{3} = 0 \) is

A. \( \frac{\pi}{3} \)
B. \( \frac{2\pi}{3} \)
C. \( \frac{4\pi}{3} \)
D. \( -\frac{\pi}{3} \)

14. If the point \( A\left(\frac{\pi}{6}, -2\right) \) lies on the graph of \( f(x) = \sin (x + \pi) - d \), then the value of \( d \) is

A. \( \frac{5}{2} \)
B. \( \frac{3}{2} \)
C. \( \frac{2 - \sqrt{3}}{2} \)
D. \( \frac{2 - \sqrt{2}}{2} \)

15. The solution to the equation \( \cot \theta = -\sqrt{3} \), \( 180^\circ < \theta < 360^\circ \), to the nearest degree, is _____.
Answer Key

1. a) $-\frac{1}{2}$  
   b) $-\sqrt{3}$  
   c) $\frac{\sqrt{2}}{2}$  
   d) $-\frac{1}{2}$  
   e) $\frac{1}{2}$  
   f) $-\sqrt{3}$

2. a) $-\frac{\sqrt{3}}{2}$  
   b) $\frac{\sqrt{3}}{3}$  
   c) $-\frac{1}{2}$  
   d) $-\frac{1}{2}$  
   e) 3  
   f) $\frac{1}{2}$

3. a) 2  
   b) $-\sqrt{3}$  
   c) $\frac{2\sqrt{3}}{3}$  
   d) $\sqrt{3}$  
   e) undefined  
   f) undefined

4. a) $\frac{1}{2}$  
   b) 1  
   c) 2  
   d) $-1$  
   e) $-\frac{2\sqrt{3}}{3}$  
   f) $-\sqrt{3}$  
   g) 2  
   h) $-\sqrt{2}$  
   i) undefined  
   j) $-\sqrt{3}$

5. a) (0,1)  
   b) (-0.22, 0.97)  
   c) (0.96, -0.28)

6. $152^\circ$  
7. $210^\circ$ or $\frac{7\pi}{6}$  
8. $\frac{2\pi}{3}$ units

9. a) $-\frac{1}{2}$  
   b) 0

10. a) $90^\circ$  
    b) $180^\circ$  
    c) $45^\circ, 135^\circ, 225^\circ, 315^\circ$  
    d) $45^\circ, 315^\circ$

11. a) $\frac{4\pi}{3}, \frac{5\pi}{3}$  
    b) $0, \pi, 2\pi$  
    c) $\frac{2\pi}{3}, \frac{4\pi}{3}$  
    d) $\frac{\pi}{4}, \frac{3\pi}{4}$  
    e) $\frac{3\pi}{4}, \frac{7\pi}{4}$  
    f) $0, \pi, 2\pi$

12. $\cot \theta = -1$  
    $\csc \theta = -\sqrt{2}$

13. B  
14. B  
15. 330
A periodic function is a function whose graph repeats regularly over some interval of the domain. The length of this interval is called the period of the function.

The amplitude of a periodic function is defined as half the distance between the maximum and minimum values of the function.

Determine the period and the amplitude for each of the following periodic functions.

Graphing the Primary Trigonometric Functions

In this lesson we will learn the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$.

To investigate the graphs of trigonometric functions we will use the table of values method to graph each function on a domain of $0^\circ \leq x \leq 360^\circ$ and use the graphing calculator to complete the graph.

We will develop the graph of $y = \sin x$ as a class lesson and the graphs of $y = \cos x$ and $y = \tan x$ will be assignment questions.
Exploration  Graphing $y = \sin x$ where $x$ is in Degrees

a) Complete the following table of values for domain $0^\circ \leq x \leq 360^\circ$. Give your answers to two decimal places where necessary.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0°</th>
<th>30°</th>
<th>60°</th>
<th>90°</th>
<th>120°</th>
<th>150°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \sin x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>210°</th>
<th>240°</th>
<th>270°</th>
<th>300°</th>
<th>330°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \sin x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Plot the points on the grid below. Do not join the points.

c) Graph $y = \sin x$ on your calculator using degree mode and the following window format.
   $x$: $[-360, 540, 30]$
   $y$: $[-1.2, 1.2, 0.2]$

d) On the grid, copy the graph from c) to complete the graph of $y = \sin x$, $-360^\circ \leq x \leq 540^\circ$.

Note • $y = \sin x$ is a periodic function whose graph continues indefinitely to the left and to the right.
• The graph shown in b) is a partial graph of $y = \sin x$ on a restricted domain.

Class Ex. #2
State the following for the function $y = \sin x$ where $x$ is in degree measure.

a) Domain ___________________ b) Range _________________

c) Amplitude _________________ d) Period _________________

e) $x$-intercept(s)______________ f) $y$-intercept(s)______________
A student was asked to reproduce the graph in Exploration b) in radian mode.

a) Write down the graphing calculator window format the student should use.

b) Sketch the graph of \( y = \sin x \) for \( 0 \leq x \leq 2\pi \) showing the intercepts.

---

**Complete Assignment Questions #1 - #12**

### Assignment

1. a) Complete the following table of values for domain \( 0 \leq x \leq 2\pi \). Give your answers to two decimal places where necessary.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \frac{2\pi}{3} )</th>
<th>( \frac{5\pi}{6} )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \cos x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{7\pi}{6} )</th>
<th>( \frac{4\pi}{3} )</th>
<th>( \frac{3\pi}{2} )</th>
<th>( \frac{5\pi}{3} )</th>
<th>( \frac{11\pi}{6} )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \cos x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Plot the points on the grid below. Do not join the points.

![Graph of \( y = \cos x \)](image)

c) Graph \( y = \cos x \) on your calculator using radian mode and the following window format.

\[
x: [-2\pi, 3\pi, \frac{\pi}{6}]
\]
\[
y: [-1.2, 1.2, 0.2]
\]

d) On the grid, copy the graph from c) to complete the graph of \( y = \cos x, \ -2\pi \leq x \leq 3\pi \).
2. State the following for the function \( y = \cos x \) where \( x \) is in radian measure.

   a) Domain ______________
   b) Range ______________
   c) Amplitude ______________
   d) Period ______________
   e) \( x \)-intercept(s) ______________
   f) \( y \)-intercept(s) ______________

3. A student was asked to reproduce one complete cycle of the graph of \( y = \cos x \) starting from zero degrees.

   a) Write down a graphing calculator window format the student could use.

   b) Sketch the graph of \( y = \cos x, \ 0^\circ \leq x \leq 360^\circ \), showing the intercepts.

4. Complete the following:

   a) When \( \sin x \) has a maximum value, the value of \( \cos x \) is ________.
   b) When \( \sin x \) has a minimum value, the value of \( \cos x \) is ________.
   c) When \( \sin x \) has a value of zero, the value of \( \cos x \) is ________________.

5. a) Using the same grid sketch the graph of \( y = \sin x \) and \( y = \cos x \) for domain \(-\pi \leq x \leq 2\pi\).

   b) For what values of \( x \), in the domain \( 0 \leq x \leq 2\pi \), do \( \sin x \) and \( \cos x \) have the same value?

   c) What is the minimum horizontal translation applied to the graph of \( y = \cos x \) which would result in the graph of \( y = \sin x \)?

   d) If \( \sin x = \cos(x - c) \) for \( x \in \mathbb{R} \), find the smallest positive value of \( c \).

   e) If \( \cos x = \sin(x - k) \) for \( x \in \mathbb{R} \), find a value for \( k \).
6. a) Complete the following table of values for domain $0^\circ \leq x \leq 180^\circ$. Give your answers to two decimal places where necessary.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0°</th>
<th>15°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>75°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \tan x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>105°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>165°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \tan x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Plot the points on the grid below. Do not join the points.

![Graphing Primary Trigonometric Functions](image)

c) To investigate what happens to the graph of $y = \tan x$ as $x$ approaches 90°, complete the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>80°</th>
<th>85°</th>
<th>89°</th>
<th>89.9°</th>
<th>89.99°</th>
<th>89.999°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \tan x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>100°</th>
<th>95°</th>
<th>91°</th>
<th>90.1°</th>
<th>90.01°</th>
<th>90.001°</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \tan x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
To graph \( y = \tan x \) on your calculator using degree mode use the following instructions:

Step 1: Enter \( y = \tan x \) using the \( Y= \) key.

Step 2: Press the \( \text{Zoom} \) key.

Step 3: Access “ZTrig” and press \( \text{Enter} \).

e) Write down the graphing calculator window format for “ZTrig”.

f) On the grid in b) complete the graph of \( y = \tan x \) for domain \(-360^\circ \leq x \leq 360^\circ\).

g) Is \( y = \tan x \) a periodic function? If so what is the period?

h) Does the concept of amplitude apply to the graph of \( y = \tan x \)?

7. State the following for the function \( y = \tan x \), \( 0^\circ \leq x \leq 360^\circ \).

\[
\begin{align*}
a) \text{Domain} & \quad \text{b) Range} \\
c) \text{Period} & \quad \text{d) } x\text{-intercept(s)} \\
e) \text{y-intercept(s)} & \\
f) \text{Equations of vertical asymptotes} \\
\end{align*}
\]

Multiple Choice

Questions #8 - #11 refer to the graph of the function \( f(x) = \tan x \), \( x \in \mathbb{R} \)

8. The domain of the function is

\[
\begin{align*}
A. & \quad x \neq n\pi, \ n \in \mathbb{I} \\
B. & \quad x \neq \frac{n\pi}{2}, \ n \in \mathbb{I} \\
C. & \quad x \neq \frac{\pi}{2} + n\pi, \ n \in \mathbb{I} \\
D. & \quad x \in \mathbb{R} \\
\end{align*}
\]

9. The asymptotes have equations

\[
\begin{align*}
A. & \quad x = n\pi, \ n \in \mathbb{I} \\
B. & \quad x = \frac{n\pi}{2}, \ n \in \mathbb{I} \\
C. & \quad x = \frac{\pi}{2} + n\pi, \ n \in \mathbb{I} \\
D. & \quad x = 2n\pi, \ n \in \mathbb{I} \\
\end{align*}
\]

10. The range of the function is

\[
\begin{align*}
A. & \quad x \neq \frac{\pi}{2} + n\pi, \ n \in \mathbb{I} \\
B. & \quad x \in \mathbb{R} \\
C. & \quad f(x) \neq \frac{\pi}{2} + n\pi, \ n \in \mathbb{I} \\
D. & \quad f(x) \in \mathbb{R} \\
\end{align*}
\]

11. The \( x\)-intercepts of the graph are

\[
\begin{align*}
A. & \quad x = n\pi, \ n \in \mathbb{I} \\
B. & \quad x = \frac{n\pi}{2}, \ n \in \mathbb{I} \\
C. & \quad x = \frac{\pi}{2} + n\pi, \ n \in \mathbb{I} \\
D. & \quad x = 2n\pi, \ n \in \mathbb{I} \\
\end{align*}
\]
12. Which of the following statements is incorrect?

A. The graph of \( y = \sin x \) has an \( x \)-intercept of \( \pi \).

B. The graph of \( y = \cos x \) has a minimum value when \( x = \pi \).

C. The graph of \( y = \tan x \) has an \( x \)-intercept of \( \pi \).

D. The graph of \( y = \cos x \) has an \( x \)-intercept of \( \pi \).

**Answer Key**

1. a)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \frac{2\pi}{3} )</th>
<th>( \frac{5\pi}{6} )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \cos x )</td>
<td>1</td>
<td>0.87</td>
<td>0.5</td>
<td>0</td>
<td>-0.5</td>
<td>-0.87</td>
<td>-1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{7\pi}{6} )</th>
<th>( \frac{4\pi}{3} )</th>
<th>( \frac{3\pi}{2} )</th>
<th>( \frac{5\pi}{3} )</th>
<th>( \frac{11\pi}{6} )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \cos x )</td>
<td>-0.87</td>
<td>-0.5</td>
<td>0</td>
<td>0.5</td>
<td>0.87</td>
<td>1</td>
</tr>
</tbody>
</table>

d)

2. a) \( x \in \mathbb{R} \)  b) \( \{ y \mid -1 \leq y \leq 1, y \in \mathbb{R} \} \)  c) \( 1 \)  d) \( 2\pi \)  e) \( \frac{\pi}{2} + n\pi, n \in \mathbb{I} \)  f) \( 1 \)

3. a) \( x \in [0, 360, 30] \)  y: \([-1.2, 1.2, 0.2]\) answers may vary  b)

4. a) \( 0 \)  b) \( 0 \)  c) \( \pm 1 \)

5. a) \( \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \)

b) \( \frac{\pi}{4} \) and \( \frac{5\pi}{4} \)

c) \( \frac{\pi}{2} \) units to the right

d) \( \frac{\pi}{2} \)  e) \( -\frac{\pi}{2} \)

---

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Absolute Value Publications
6. a) 

<table>
<thead>
<tr>
<th>x</th>
<th>0°</th>
<th>15°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>75°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>y = tan x</td>
<td>0</td>
<td>0.27</td>
<td>0.58</td>
<td>1</td>
<td>1.73</td>
<td>3.73</td>
<td>undefined</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>105°</th>
<th>120°</th>
<th>135°</th>
<th>150°</th>
<th>165°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>y = tan x</td>
<td>-3.73</td>
<td>-1.73</td>
<td>-1</td>
<td>-0.58</td>
<td>-0.27</td>
<td>0</td>
</tr>
</tbody>
</table>

c) 

<table>
<thead>
<tr>
<th>x</th>
<th>80°</th>
<th>85°</th>
<th>89°</th>
<th>89.9°</th>
<th>89.99°</th>
<th>89.999°</th>
</tr>
</thead>
<tbody>
<tr>
<td>y = tan x</td>
<td>5.67</td>
<td>11.43</td>
<td>57.29</td>
<td>573</td>
<td>5730</td>
<td>57296</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>100°</th>
<th>95°</th>
<th>91°</th>
<th>90.1°</th>
<th>90.01°</th>
<th>90.001°</th>
</tr>
</thead>
<tbody>
<tr>
<td>y = tan x</td>
<td>-5.67</td>
<td>-11.43</td>
<td>-57.29</td>
<td>-573</td>
<td>-5730</td>
<td>-57296</td>
</tr>
</tbody>
</table>

e) \(x: [-352.5, 352.5, 90]\) \(y: [-4, 4, 1]\)  
f) yes, the period is 180°  
h) no  

7. a) \(\{x | x \neq 90°, 270° \ x \in \mathbb{R}\}\)  
b) \(y \in \mathbb{R}\)  
c) 180°  
d) \(0°, 180°, 360°\)  
e) 0  
f) \(x = 90°, x = 270°\)

8. C  
9. C  
10. D  
11. A  
12. D
Warm-Up

Introduction

In the next two lessons we will consider the graphs of the functions whose equations are

\[ y = a \sin[b(x - c)] + d \quad \text{and} \quad y = a \cos[b(x - c)] + d \]

and relate them to the graphs of the functions whose equations are \( y = \sin x \) and \( y = \cos x \).

In this lesson we concentrate on the effects of the parameters \( a \) and \( b \).

Class Ex. #1

a) Use the knowledge gained from the transformation unit to describe how the graph of the given function compares to the graph of \( y = \sin x \), where \( x \) is in degrees.

i) \( y = 2 \sin x \)

ii) \( y = \sin 2x \)

iii) \( y = -3 \sin x \)

iv) \( y = \sin (-3x) \)

b) In which of the above examples is there a change in amplitude compared to the graph of \( y = \sin x \)?

c) In which of the above examples is there a change in period compared to the graph of \( y = \sin x \)?

d) Complete the table. Use a graphing calculator if necessary.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Amplitude</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sin x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = 2 \sin x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \sin 2x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = -3 \sin x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \sin (-3x) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = 5 \sin 4x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{1}{3} \sin \frac{1}{2}x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = a \sin bx )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

e) Describe the effect of the parameter “\( a \)” on the graphs of \( y = a \sin x \).

f) Describe the effect of the parameter “\( b \)” on the graphs of \( y = \sin bx \).

g) Would you expect similar effects on the graph of \( y = a \cos bx \)? Investigate if necessary.
Effects of $a$ and $b$ in $y = a \sin bx$, $y = a \cos bx$, and $y = a \tan bx$

Changing the parameter “$a$” on the graphs of $y = a \sin x$ and $y = a \cos x$ results in a vertical stretch about the $x$-axis with the following:

- an expansion if $a > 1$
- a compression if $0 < a < 1$
- if $a < 0$, the result is a reflection in the $x$-axis and a vertical stretch of factor $a$ about the $x$-axis.

Changing the parameter “$a$” on the graph of $y = a \tan x$ also results in a vertical stretch of factor “$a$” about the $x$-axis.

Changing the parameter “$b$” on the graphs of $y = \sin bx$, $y = \cos bx$, and $y = \tan bx$ results in a horizontal stretch about the $y$-axis with the following:

- a compression if $b > 1$
- an expansion if $0 < b < 1$
- if $b < 0$, the result is a reflection in the $y$-axis and a horizontal stretch of factor $\frac{1}{b}$ about the $y$-axis.

$y = a \sin bx \quad$ or $\quad y = a \cos bx \quad$ $y = a \tan bx$

<table>
<thead>
<tr>
<th>Function</th>
<th>Amplitude</th>
<th>Period (Degree)</th>
<th>Period (Radian)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = a \sin bx$</td>
<td>$</td>
<td>a</td>
<td>\left(\frac{\text{Max} - \text{Min}}{2}\right)$</td>
</tr>
<tr>
<td>$y = a \cos bx$</td>
<td>$</td>
<td>a</td>
<td>\left(\frac{\text{Max} - \text{Min}}{2}\right)$</td>
</tr>
<tr>
<td>$y = a \tan bx$</td>
<td>No applicable</td>
<td>$\frac{180^\circ}{</td>
<td>b</td>
</tr>
</tbody>
</table>

Hints for Graphing a Trigonometric Function Manually

- Sketch the primary trigonometric graph, i.e. $y = \sin x$ or $y = \cos x$.
- Adjust the basic graph for any change in amplitude by considering the max and min points.
- Adjust the new graph for any change in period by dividing the period into four parts using the maximum and minimum points, and the points where the graph intersects the mid-line (the horizontal line running through the centre of the graph).
Consider the graph of \( y = 4 \cos 2x \), \( 0 \leq x \leq 2\pi \).

**a)** State the amplitude and period.

**b)** Sketch the graph on the grid. Use a graphing calculator to verify.

---

Write the equation of:

**a)** a sine function having an amplitude of \( \frac{2}{3} \) and a period of \( \frac{\pi}{6} \).

**b)** a cosine function having an amplitude of 3 and a period of 720°.

**c)** a tangent function having a period of \( \frac{\pi}{2} \).

---

Consider the graph shown.

**a)** State the amplitude and period.

**b)** Write the equation of the sine function which the graph represents.
The graph represents the effect of tides on mean sea level over a 24 hour period. The graph has equation \( h(t) = a \cos bt \), where \( t \) is in hours and \( h \) is the height, in metres, relative to mean sea level. Determine the equation of the graph.

**Complete Assignment Questions #1 - #16**

**Assignment**

1. Describe how the graph of the given function compares to the graph of \( y = \cos x \).
   a) \( y = 5 \cos x \)
   b) \( y = 2 \cos \frac{1}{2} x \)
   c) \( y = -\frac{1}{3} \cos 4x \)
   d) \( y = 0.2 \cos (-6x) \)

2. State the amplitude.
   a) \( y = 5 \sin x \)   b) \( y = \cos 3x \)   c) \( y = \frac{7}{3} \sin 2x \)   d) \( y = -4 \cos \frac{5}{6} \theta \)

3. State the period in degrees.
   a) \( y = 6 \sin x \)   b) \( y = \tan 3x \)   c) \( y = \frac{2}{3} \cos \frac{x}{7} \)   d) \( y = -2 \tan \frac{2}{3} \theta \)

4. State the period in radians.
   a) \( y = 7 \tan x \)   b) \( y = \cos 3x \)   c) \( y = \frac{1}{4} \sin \frac{x}{3} \)   d) \( y = 5 \tan \frac{1}{2} \theta \)
5. Write the equation of a \textbf{sine} function having the indicated amplitude and period.

a) amplitude 2, period 1080°  b) amplitude 8, period $\frac{\pi}{4}$  c) amplitude $\frac{3}{2}$, period $6\pi$

6. Write the equation of a \textbf{cosine} function having the indicated amplitude and period.

a) amplitude 1, period 180°  b) amplitude 5, period $\frac{4\pi}{3}$  c) amplitude $\frac{5}{3}$, period $3\pi$

7. Write the equation of a \textbf{tangent} function having the indicated period.

a) period 45°  b) period $\frac{4\pi}{3}$

8. Determine the equation of each graph in the form:

a) $y = a \sin bx$

b) $y = a \cos bx$
9. The trigonometric graph shown has a maximum value of 3 and a minimum value of \(-3\). Determine the equation of the graph in the form \(y = a \sin bx\).

10. The trigonometric graph shown has a maximum value of 5 and a minimum value of \(-5\). Determine the equation of the graph in the form \(y = a \cos bx\).

11. Consider the graph of \(y = 3 \cos \frac{x}{2}, 0 \leq x \leq 2\pi\).
   
   a) State the amplitude and period.

   b) Sketch the graph on the grid. Use a graphing calculator to verify.
12. The graph represents the change in sea level over a 24 hour period. The graph has equation \( h(t) = a \sin bt \), where \( t \) is in hours and \( h \) is the height, in metres, relative to mean sea level.

a) Determine the equation of the graph.

b) Calculate the height above mean sea level, to the nearest tenth, when \( t = 4 \).

13. a) Which transformations applied to the graph of \( y = \sin x \) would result in the graph shown?

b) Write the equation of the graph in the form \( y = a \sin bx \).

14. Which of the following functions does not have a period of \( \pi \)?
   A. \( y = \sin 2x \)   B. \( y = \cos 2x \)   C. \( y = \tan 2x \)   D. \( y = \tan x \)

15. Which of the following statements is incorrect?
   A. The maximum value of the graph of \( y = 3 \cos 2x \) is 3.
   B. The graph of \( y = 3 \sin 2x \) has a \( y \)-intercept of 3.
   C. The graph of \( y = 4 \cos 3x \) has an \( x \)-intercept of \( \frac{\pi}{6} \).
   D. The graph of \( y = 2 \tan 2x \) has an asymptote with equation \( x = \frac{\pi}{4} \).
The graph shown has equation $y = \tan bx$.

The value of $b$, to the nearest tenth, is _____.

**Answer Key**

1. a) a vertical expansion by a factor of 5 about the $x$-axis.
   b) a vertical expansion by a factor of 2 about the $x$-axis and a horizontal expansion by a factor of 2 about the $y$-axis.
   c) a vertical compression by a factor of $\frac{1}{3}$ about the $x$-axis, a horizontal compression by a factor of $\frac{1}{5}$ about the $y$-axis, and a reflection in the $x$-axis.
   d) a vertical compression by a factor of 0.2 about the $x$-axis, a horizontal compression by a factor of $\frac{1}{6}$ about the $y$-axis, and a reflection in the $y$-axis.

2. a) 5 b) 1 c) $\frac{7}{3}$ d) 4  
3. a) $360^\circ$ b) $60^\circ$ c) $2520^\circ$ d) $270^\circ$

4. a) $\pi$ b) $\frac{2\pi}{3}$ c) $6\pi$ d) $2\pi$

5. a) $y = 2 \sin \frac{1}{3}x$  
b) $y = 8 \sin 8x$  
c) $y = \frac{3}{2} \sin \frac{1}{3}x$

6. a) $y = \cos 2x$  
b) $y = 5 \cos \frac{3}{2}x$  
c) $y = \frac{5}{3} \cos \frac{2}{3}x$

7. a) $y = \tan 4x$  
b) $y = \tan \frac{3}{4}x$

8. a) $y = 15 \sin \frac{1}{3}x$  
b) $y = 4 \cos \frac{1}{4}x$

9. $y = 3 \sin 2x$

10. $y = 5 \cos 9x$

11. a) amp = 3, period = $4\pi$  
b) $\rightarrow$  

12. a) $y = 8 \sin \frac{\pi}{6}t$  
b) 6.9 metres

13. a) a vertical expansion by a factor of 2 about the $x$-axis, a horizontal compression by a factor of $\frac{1}{3}$ about the $y$-axis, a reflection in the $x$- or $y$-axis.
   b) $y = -2 \sin 3x$ or $y = 2 \sin (-3x)$

14. C  
15. B  
16. 0.5
Trigonometry - Functions and Graphs Lesson #9:
Transformations of Trigonometric Functions Part 2

Warm-Up

Introduction

In this lesson we will consider the graphs of the functions whose equations are

\[ y = a \sin[b(x - c)] + d \quad \text{and} \quad y = a \cos[b(x - c)] + d \]

and relate them to the graphs of the functions whose equations are \( y = \sin x \) and \( y = \cos x \).

In the first part of the lesson we concentrate on the effects of the parameters \( c \) and \( d \).

Class Ex. #1

a) Describe how the graph of the given function compares to the graph of \( y = \sin x \), where \( x \) is in degrees

i) \( y = \sin(x - 30^\circ) \)

ii) \( y = \sin x + 2 \)

iii) \( y = \sin(x + 60^\circ) - 1 \)

iv) \( y - 45 = \sin(x - 45^\circ) \)

In trigonometry:
- a horizontal translation is called a **horizontal phase shift**, and,
- a vertical translation is called a **vertical displacement**.

Class Ex. #2

Complete the table to describe how the graph of the given function compares to the graph of \( y = \sin x \) where \( x \) is in radians. Use a graphing calculator if necessary.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Horizontal Phase Shift</th>
<th>Vertical Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sin x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \sin \left( x - \frac{\pi}{4} \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \sin x + 5 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y + \pi = \sin \left( x + \frac{3\pi}{2} \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \sin(x - c) + d )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = a \sin[b(x - c)] + d )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Would you expect similar effects on the graph of \( y = a \cos[b(x - c)] + d \)? Investigate if necessary.
**Effects of c and d in** \( y = a \sin [b(x - c)] + d \) **and** \( y = a \cos [b(x - c)] + d \)

Changing the parameter “c” on the graphs of

\[ y = a \sin [b(x - c)] + d \] and \[ y = a \cos [b(x - c)] + d \]

results in a horizontal phase shift with the following:

- a horizontal phase shift to the right if \( c > 0 \)
- a horizontal phase shift to the left if \( c < 0 \)

Changing the parameter “d” on the graphs of

\[ y = a \sin [b(x - c)] + d \] and \[ y = a \cos [b(x - c)] + d \]

results in a vertical displacement with the following:

- a vertical displacement up if \( d > 0 \)
- a vertical displacement down if \( d < 0 \)

The vertical displacement can be determined from a graph using the formula \( d = \frac{\text{Max} + \text{Min}}{2} \).

**Summary of the Effects of the Parameters a, b, c, and d**

For \[ y = a \sin [b(x - c)] + d \]

- \( \text{amplitude} = |a| = \frac{\text{Max} - \text{Min}}{2} \)
- \( \text{period} = \frac{360^\circ}{|b|} \) (for degree measure)
- \( \text{period} = \frac{2\pi}{|b|} \) (for radian measure)
- horizontal phase shift = \( c \)
  - to the right if \( c > 0 \)
  - to the left if \( c < 0 \)
- vertical displacement = \( d \)
  - up if \( d > 0 \)
  - down if \( d < 0 \)
  - \( d = \frac{\text{Max} + \text{Min}}{2} \)

For \[ y = a \tan[b(x - c)] + d \]

- \( \text{amplitude} - \text{not applicable} \)
- \( a \) value represents
  - a vertical expansion or,
  - a vertical compression
- \( \text{period} = \frac{180^\circ}{|b|} \) (for degree measure)
- \( \text{period} = \frac{\pi}{|b|} \) (for radian measure)
- horizontal phase shift = \( c \)
  - to the right if \( c > 0 \)
  - to the left if \( c < 0 \)
- vertical displacement = \( d \)
  - up if \( d > 0 \)
  - down if \( d < 0 \)
Consider equations of the form \( y = a \sin[b(x - c)] + d \) and \( y = a \cos[b(x - c)] + d \), where \( a = 1 \), and \( b = 1 \). Write the equation which represents;

a) a cosine function having a horizontal phase shift of 75° right

b) a sine function having a horizontal phase shift of \( \frac{3\pi}{5} \) radians left, and a vertical displacement 4 units up

Find the amplitude, period, horizontal phase shift, and vertical displacement of the graphs of the following functions defined on \( x \in \mathbb{R} \).

a) \( y = 2 \sin(3(x + \pi)) - 4 \)  
b) \( y = -\frac{2}{3} \cos \frac{1}{4}(x - \frac{\pi}{12}) + 3 \)

c) Compare the answer to Class Ex. #4a and Class Ex. #5a.
The graphs from a) - d) represent the same trigonometric function.

a) Write the equation of the graph in the form $y = a \sin (x - c)$ if $a > 0$ and there is a minimum possible horizontal phase shift.

b) Write the equation of the graph in the form $y = a \sin (x - c)$ if $a < 0$ and there is a minimum possible horizontal phase shift.

c) Write the equation of the graph in the form $y = a \cos (x - c)$ if $a > 0$ and there is a minimum possible horizontal phase shift.

d) Write the equation of the graph in the form $y = a \cos (x - c)$ if $a < 0$ and there is a minimum possible horizontal phase shift.
Consider the graph shown.

a) If the graph represents a sine function where \( a > 0 \), write the equation represented by the graph.

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Period</th>
<th>Min Horizontal Shift</th>
<th>Vertical Displacement</th>
<th>Equation</th>
</tr>
</thead>
</table>

b) If the graph in a) represents a sine function where \( a < 0 \), write the equation represented by the graph.

c) If the graph represents a cosine function where \( a > 0 \), write the equation represented by the graph.

<table>
<thead>
<tr>
<th>Amplitude</th>
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<th>Min Horizontal Shift</th>
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<th>Equation</th>
</tr>
</thead>
</table>

d) If the graph in c) represents a cosine function where \( a < 0 \), write the equation represented by the graph.
Consider the graphs of the functions \( y = a \sin [b(x - c)] + d \) and \( y = a \cos [b(x - c)] + d \).

a) Changing which of the parameters \( a, b, c \) and \( d \) affect the:
   i) domain  ii) range  iii) amplitude  iv) period  v) zeros

b) State the maximum and minimum values of the functions in terms of \( a, b, c, \) and \( d \), if \( a > 0 \).

c) Determine the range of the function \( y = 3 \sin 2(x - \pi) - 4 \).

**Complete Assignment Questions #3 - #12**

**Assignment**

1. Determine the amplitude, period, horizontal phase shift, and the vertical displacement for each function.

   a) \( y = \cos \left( x - \frac{\pi}{4} \right) + 3 \)  
   b) \( y = 3 \cos \left( \frac{1}{2}x - \frac{\pi}{2} \right) \)  
   c) \( y = 3 \cos \left( \frac{1}{2}x - \frac{\pi}{2} \right) \)

   d) \( y = \sin \left( 4x - \frac{\pi}{2} \right) \)  
   e) \( y = -2 \cos (3x - 45^\circ) + 4 \)  
   f) \( y = 7 \sin \left( \frac{1}{4}x + 20^\circ \right) - 1 \)
2. a) Find the equation of a sine function that has a vertical displacement 3 units up, a horizontal phase shift of 60° to the left, a period of 210° and an amplitude of 4.

b) Find the equation of a cosine function that has a vertical displacement 5 units down, a horizontal phase shift of \( \frac{2\pi}{3} \) radians to the right, a period of \( \frac{5\pi}{4} \) and an amplitude of 3.

3. Graphs 1 and 2 each represent the graphs of trigonometric functions.

a) Assuming a minimum possible phase shift, write the equation of each graph in the form 
\[ y = a \sin [b(x - c)] + d \] if:

i) \( a > 0 \)

ii) \( a < 0 \)
b) Assuming a minimum possible phase shift, write the equation of each graph in the form \( y = a \cos \left[ b(x - c) \right] + d \) if:

i) \( a > 0 \)  
ii) \( a < 0 \)

4. The cosine graph shown has a range \(-3 \leq y \leq 9\). The graph has an equation in the form \( y = a \cos \left[ b(x - c) \right] + d, \ a > 0 \). Determine the equation if the graph has a minimum possible phase shift.
5. The sine graph shown has a maximum value of 20 and a minimum value of 10. The graph has an equation in the form \( y = a \sin [b(x - c)] + d \) with \( a > 0 \). Determine the equation if the graph has a minimum possible phase shift.

6. The graphs shown have an equation in the form \( y = \tan b(x - c) \). Determine the equation of each graph.

7. Determine the range of the functions represented below.
   a) \( y = 2 \sin x - 2 \)  
   b) \( y = 3 \cos \frac{1}{2}(x - \pi) + 1 \)  
   c) \( y = -\frac{1}{2} \cos 4(x - \pi) - 3 \)  
   d) \( y = a \sin [b(x - c)] + d \)

8. Which of the following graphs has the same \( x \)-intercepts as the graph of \( y = \cos x \)?
   A. \( y = \cos 4x \)  
   B. \( y = 4 \cos x \)  
   C. \( y = \cos x + 4 \)  
   D. \( y = \cos (x + 4) \)

9. Which equation is a tangent function with period \( \frac{\pi}{3} \), and a vertical displacement –3?
   A. \( y = \tan \frac{\pi}{3}x - 3 \)  
   B. \( y = \tan 3(x - 3) \)  
   C. \( y = \tan 3x - 3 \)  
   D. \( y = \tan 6x - 3 \)
10. The equation \( y = \pi \cos (\pi x - \pi) \) has a period and a horizontal phase shift to the right, respectively, of

A. \( \pi \) and \( \pi \)
B. \( \pi \) and 1
C. 2 and \( \pi \)
D. 2 and 1

11. Which statement concerning the graph of \( y = -\cos \frac{x}{2} + 2 \) is not correct?

A. The maximum value is 6.
B. The period is \( 4\pi \).
C. The amplitude is \(-4\).
D. The vertical displacement is 2.

12. The period, to the nearest tenth, of the function \( y = \sin 0.25x \), where \( x \) is in radians, is ______.
**Warm-Up Review of Reciprocal Functions**

Recall the properties of reciprocal functions by completing the following:

1. • When \( f(x) = 0 \), the graph of \( y = \frac{1}{f(x)} \) may have a ____________ ____________.
   • When \( f(x) \) is positive, \( \frac{1}{f(x)} \) is ____________.
   • When \( f(x) \) is negative, \( -\frac{1}{f(x)} \) is ____________.

2. • When \( f(x) = 1 \), \( \frac{1}{f(x)} = \) ____ . When \( f(x) = -1 \), \( \frac{1}{f(x)} = \) ____.
   • The invariant points for a reciprocal transformation can be found where the lines \( y = \pm 1 \) intersect the graphs of \( f(x) \) and \( \frac{1}{f(x)} \).

3. • When \( f(x) \) increases over an interval, \( \frac{1}{f(x)} \) ____________ over the same interval.
   • When \( f(x) \) decreases over an interval, \( \frac{1}{f(x)} \) ____________ over the same interval.

4. • When \( f(x) \) approaches zero, \( \frac{1}{f(x)} \) approaches \( \pm \infty \) and the graph of \( \frac{1}{f(x)} \) approaches a ____________ asymptote.
   • When \( f(x) \) approaches \( \pm \infty \), \( \frac{1}{f(x)} \) approaches zero and the graph of \( \frac{1}{f(x)} \) approaches a ____________ asymptote.

• Remember: \( \sin^{-1} x \) does NOT mean \( \frac{1}{\sin x} \). \( \sin^{-1} x \) represents the inverse of the function \( \sin x \). The reciprocal of \( \sin x \) is \( \csc x \).

• The above properties can be used as a general aid to sketch the reciprocal trigonometric functions.

**Sketching the Graph of a Reciprocal Trigonometric Function**

Use the following general procedure to sketch the graph of a reciprocal trigonometric function.

1. Sketch the vertical asymptotes.

2. Mark the invariant points.

3. Where \( y \) approaches zero on the original graph, \( y \) approaches positive or negative infinity on the reciprocal graph.
The graph of \(y = \sin x, \ -2\pi \leq x \leq 2\pi\) is shown.

a) Graph \(y = \csc x\), the reciprocal of \(y = \sin x\), using the properties on the previous page.

b) State the equations of the asymptotes.

c) List the invariant points.

d) Complete the table for \(x \in \mathbb{R}\).

<table>
<thead>
<tr>
<th>Function</th>
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</thead>
<tbody>
<tr>
<td>(y = \sin x)</td>
<td></td>
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<tr>
<td>(y = \csc x)</td>
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</table>

Use a graphing calculator to:

a) Graph \(y = \csc x\) and \(y = \csc 2x\).

b) State an appropriate graphing calculator window where \(x\) is in radians.

c) Complete the table for \(x \in \mathbb{R}\).

<table>
<thead>
<tr>
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</tr>
<tr>
<td>(y = \csc 2x)</td>
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</table>

d) Complete the following statements based on your observations in a), b), c).

i) The graph of \(y = \csc 2x\) is a transformation of the graph of \(y = \csc x\) by a \___________ \___________\ by a factor of _____ about the ___ - axis.

ii) Compared to the asymptotes of \(y = \csc x\), the asymptotes of the graph of \(y = \csc 2x\) are \___________\ as frequent.
Warm-Up #2  Review of Absolute Value Functions

Recall the properties of absolute value functions by completing the following:

- When \( f(x) \geq 0 \), (i.e. the graph of \( y = f(x) \) is above the \( x \)-axis), the graph of \( y = |f(x)| \) is ____________ to the graph of \( y = f(x) \).

- When \( f(x) \leq 0 \), (i.e. the graph of \( y = f(x) \) is below the \( x \)-axis), the graph of \( y = |f(x)| \) is a ____________ of the graph of \( y = f(x) \) in the \( x \)-axis.

The graph of \( y = \cos x \), \(-2\pi \leq x \leq 2\pi\) is shown.

**a)** Sketch the graph of \( y = |\cos x| \)

**b)** State the domain and range of \( y = |\cos x| \).

Complete Assignment Questions #1 - #12

Assignment

1. The graph of \( y = \cos x \), \(-2\pi \leq x \leq 2\pi\) is shown.

   a) Graph \( y = \sec x \), the reciprocal of \( y = \cos x \).

   b) State where \( \sec x \) is undefined.

   c) List the invariant points.

   d) Complete the table for \( x \in \mathbb{R} \).

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<td></td>
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<tr>
<td>( y = \sec x )</td>
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<td></td>
</tr>
</tbody>
</table>
2. The graph of $y = \tan x$, $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$ is shown.

   a) Graph $y = \cot x$, the reciprocal of $y = \tan x$.

   b) State the equations of the asymptotes of $y = \cot x$.

   c) List the invariant points.

   d) Complete the table for $x \in \mathbb{R}$.

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<td></td>
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<tr>
<td>$y = \cot x$</td>
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</tbody>
</table>

3. Use a graphing calculator to:

   a) Graph $y = \csc x$ and $y = \csc \left( x + \frac{\pi}{4} \right)$

   b) State an appropriate graphing calculator window in radians.

   c) Complete the table for $x \in \mathbb{R}$.

<table>
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</tbody>
</table>

4. Use a graphing calculator to:

   a) Graph $y = \sec x$ and $y = \sec 3x$

   b) State an appropriate graphing calculator window in radians.

   c) Complete the table for $x \in \mathbb{R}$.

<table>
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<tr>
<td>$y = \sec 3x$</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
5. The graph of \( y = 2 \sin x \), \(-2\pi \leq x \leq 2\pi\) is shown.

   a) Graph the reciprocal of \( y = 2 \sin x \).

   b) State the equation of the reciprocal function.

6. The graph of the function \( y = \sec 2x \) is shown.

   a) Graph the reciprocal of \( y = \sec 2x \).

   b) State the equation of the reciprocal function.

7. Sketch the following graphs

   a) \( y = |\sin x| \)

   b) \( y = |\tan x| \)
8. The graph represents a reciprocal trigonometric function after a single transformation. Determine the equation of each graph. Verify with a graphing calculator.

9. Which of the following describes the asymptotes for \( y = \sec x \)?

   A. \( x = n\pi, \ n \in I \)
   
   B. \( x = \frac{\pi}{2} + n\pi, \ n \in I \)
   
   C. \( x = 2n\pi, \ n \in I \)
   
   D. \( x = \frac{\pi}{2} + 2n\pi, \ n \in I \)

10. The minimum positive value of \( y \) on the graph of \( y = \csc \frac{1}{2}x \) is

    A. \( \frac{1}{2} \)
    
    B. 1
    
    C. 2
    
    D. impossible to determine
11. The graph of \( y = \sec 2x \) is a transformation of the graph of \( y = \csc 2x \) by a horizontal translation of

A. \( \frac{\pi}{2} \) radians left  
B. \( \frac{\pi}{2} \) radians right  
C. \( \frac{\pi}{4} \) radians left  
D. \( \frac{\pi}{4} \) radians right

**Numerical Response**

12. The graph of \( y = 4\sin x \) and its reciprocal are drawn. If the reciprocal graph has equation \( y = k \csc x \), then the value of \( k \) to the nearest hundredth is _____ .

13. The maximum value, to the nearest tenth, of the function \( f(x) = \mid \cos x - 2 \mid \) is _____ .

**Answer Key**

1. a) see graph below  
   b) \( x = -\frac{3\pi}{2}, -\frac{3\pi}{2}, -\frac{3\pi}{2} \)  
   c) \((-2\pi, 1), (\pi, -1), (0,1), (\pi, -1), (2\pi, 1)\)
   
   d) see table below

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<td>( y = \cos x )</td>
<td>( x \in \mathbb{R} )</td>
<td>(-1 \leq y \leq 1, y \in \mathbb{R} )</td>
</tr>
<tr>
<td>( y = \sec x )</td>
<td>( x \neq \pm \frac{\pi}{2} + n\pi, n \in \mathbb{Z}, x \in \mathbb{R} )</td>
<td>( y \leq -1 ) and ( y \geq 1, y \in \mathbb{R} ) or (</td>
</tr>
</tbody>
</table>

2. a) see graph below  
   b) \( x = \pm \pi, x = 0, x = \pi \)  
   c) \((-\frac{5\pi}{4}, -1), (-\frac{3\pi}{4}, 1), (-\frac{\pi}{4}, -1), (\frac{\pi}{4}, 1), (\frac{3\pi}{4}, -1), (\frac{5\pi}{4}, 1)\)
   
   d) See table below

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</tr>
<tr>
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<td>( x \neq n\pi, n \in \mathbb{Z}, x \in \mathbb{R} )</td>
<td>( y \in \mathbb{R} )</td>
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</table>

3. b) answers may vary  
   c) 

<table>
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<td>( 2\pi )</td>
<td>( x = n\pi, n \in \mathbb{Z} )</td>
</tr>
<tr>
<td>( y = \csc \left( x + \frac{\pi}{4} \right) )</td>
<td>( x \neq n\pi - \frac{\pi}{4}, n \in \mathbb{Z}, x \in \mathbb{R} )</td>
<td>( y \leq -1 ) and ( y \geq 1, y \in \mathbb{R} )</td>
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<td>( x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z} )</td>
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4. b) answers may vary  
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<td>$y \leq -1$ and $y \geq 1$, $y \in \mathbb{R}$</td>
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<td>$x = \frac{\pi}{2} + n\pi, n \in \mathbb{I}$</td>
</tr>
<tr>
<td>$y = \sec 3x$</td>
<td>$x \neq \frac{\pi}{6} + n\frac{\pi}{3}, n \in \mathbb{I}, x \in \mathbb{R}$</td>
<td>$y \leq -1$ and $y \geq 1$, $y \in \mathbb{R}$</td>
<td>$\frac{2\pi}{3}$</td>
<td>$x = \frac{\pi}{6} + n\frac{\pi}{3}, n \in \mathbb{I}$</td>
</tr>
</tbody>
</table>

5. a) see graph below  
b) $y = \frac{1}{2} \csc x$  
6. a) see graph below  
b) $y = \cos 2x$

7. a) see graph below  
b) see graph below

8. a) $y = \cot \left(x + \frac{3\pi}{8}\right)$  
b) $y = \sec x - 2$

9. B  
10. B  
11. C  
12. 0.25  
13. 3.0
Trigonometry -
Equations, Identities, and Modelling Lesson #1:
Solving First Degree Trigonometric Equations

Warm-Up #1
Introduction

In this lesson we will be solving first degree equations where the power of the trigonometric function is one (e.g. $2 \sin x + 1 = 0$). We will:

- review the algebraic procedure for solving a first degree equation on a domain of length $2\pi$.
- use a graphical approach to determine an approximate solution.
- find the general solution over the domain of real numbers.

Warm-Up #2
Review

Use an algebraic procedure to solve the equation $\sin x = -\frac{1}{2}, \ 0 \leq x \leq 2\pi$.

General Solution

The general solution to a trigonometric equation is the solution over the domain of real numbers.
Warm-Up #3  Exploring a General Solution Using a Graphical Approach

Consider the equation \( \sin x = -\frac{1}{2} \), (i.e. \( \sin x + \frac{1}{2} = 0 \))

a) Using the following graphical method to estimate the solution to the equation on the domain \( 0 \leq x \leq 2\pi \).

   - Use window \( x: [0, 2\pi, \frac{\pi}{6}] \) \( y: [-2, 2, 0.5] \)
   - Graph \( Y_1 = \sin x + \frac{1}{2} \)
   - Determine (in terms of \( \pi \)), the \( x \)-intercepts of graph \( Y_1 \) where \( 0 \leq x \leq 2\pi \).

   - Solution is __________________

b) Using the following graphical method to estimate the solution to the equation on the domain \( -2\pi \leq x \leq 4\pi \).

   - Use the window \( x: [-2\pi, 4\pi, \frac{\pi}{6}] \) \( y: [-2, 2, 0.5] \)
   - Graph \( Y_1 = \sin x + \frac{1}{2} \)
   - Determine the \( x \)-intercepts of graph \( Y_1 \) where \( -2\pi \leq x \leq 4\pi \). Give the answers as exact values.

   - Solution is __________________

c) Notice that the solution in b) forms two sets of answers which differ by \( 2\pi \) (or a multiple of \( 2\pi \)) from the answers in a). Complete:

   _________________ are angles which differ by \( 2\pi \) (or a multiple of \( 2\pi \))

   _________________ are angles which differ by \( 2\pi \) (or a multiple of \( 2\pi \))

d) Use this idea to write the general solution to the equation \( \sin x = -\frac{1}{2} \) where \( x \in \mathbb{R} \)

   - General Solution is __________________

e) The graph of \( y = \sin x \) is shown. Show the solution to the equation \( \sin x = -\frac{1}{2} \) on the given domain by drawing the line with equation \( y = -\frac{1}{2} \) and marking the points of intersection.
The answers in parts b), c) and d) differ by 2π radians because the graph of \( y = \sin x \) has a **period** of 2π radians.

---

**Finding a General Solution Using a Graphical Approach**

Use the following procedure to find the general solution

1. Use a graphing calculator to solve the equation where the domain is **one period** of the graph of the function (usually \( 0 \leq x \leq 2\pi \)). Use either of the following methods:
   - enter one side of the equation in \( Y_1 \) and the other side of the equation in \( Y_2 \) and find the \( x \)-coordinates of the points of intersection.
   - set the equation equal to zero and enter that into \( Y_1 \) and find the \( x \)-intercepts.

2. Determine the general solution by adding or subtracting multiples of the period of the graph of the function.

---

**Class Ex. #1**

Solve the equation \( \cos x - 0.75 = 0, \ x \in \mathbb{R} \), using two different graphical approaches. Give answers to the nearest hundredth.
General Solution Using an Algebraic Approach

Use the following procedure to find the general solution using an algebraic approach:

1. Solve the equation where the domain is one period of the graph of the function (usually $0 \leq x \leq 2\pi$).

2. The general solution can be determined by adding or subtracting multiples of the period.

Use an algebraic procedure to find the general solution to the equation $2\cos x - \sqrt{3} = 0$, $x \in \mathbb{R}$, where $x$ is in radian measure.

Complete Assignment Questions #1 - #13
Assignment

1. The diagram shows the graph of the equations $y = \cos x$ and $y = 0.5$ in $0 \leq x \leq 2\pi$.

a) **Explain** how to use the graph to determine the approximate solutions to the equation $\cos x = 0.5$, $0 \leq x \leq 2\pi$.

b) Write the solutions to the equation $\cos x = 0.5$, $0 \leq x \leq 2\pi$.
   Give solutions as exact values.

c) Write the general solution to the equation $\cos x = 0.5$.

2. The diagram shows the graph of the equation $y = \tan x - 1$ on the domain $0 \leq x \leq 2\pi$.

a) **Explain** how to use the graph to determine the approximate solutions to the equation $\tan x = 1$, $0 \leq x \leq 2\pi$.

b) Write the solutions to the equation $\tan x = 1$, $0 \leq x \leq 2\pi$.
   Give solutions as exact values.

c) Write the general solution to the equation $\tan x = 1$.

3. Determine the solution to each of the following equations, defined on the domain $0 \leq x \leq 2\pi$, using a **graphical** approach. Give solutions as exact values.

   a) $\sin x = \frac{\sqrt{3}}{2}$

   b) $\tan x = -1$

   c) $2 \sec x - 4 = 0$

4. Use the solutions in #3 to write the general solutions to the equations.

   a) $\sin x = \frac{\sqrt{3}}{2}$

   b) $\tan x = -1$

   c) $2 \sec x - 4 = 0$
5. Determine the solution (to the nearest hundredth) to each of the following equations, defined on the domain $0 \leq x \leq 2\pi$, using a graphical approach.
   a) $\cos x = 0.6$  
   b) $\cot x = -\frac{1}{2}$  
   c) $\csc x - 3 = 0$

6. Use the solutions in #5 to write the general solutions to the equations.
   a) $\cos x = 0.6$
   b) $\cot x = -\frac{1}{2}$
   c) $\csc x - 3 = 0$

7. Determine the solution to each of the following equations, defined on the domain $0 \leq x \leq 2\pi$, using an algebraic approach.
   a) $2 \sin x = -\sqrt{3}$  
   b) $\cot x + \sqrt{3} = 0$  
   c) $3 \sec x - 6 = 0$

8. Use the solutions in #7 to write the general solutions to the equations
   a) $2 \sin x = -\sqrt{3}$
   b) $\cot x + \sqrt{3} = 0$
   c) $3 \sec x - 6 = 0$

9. Use an algebraic approach to determine the general solution to the following equations where $x$ is measured in radians.
   a) $2 \cos x - \sqrt{2} = 0$  
   b) $\csc x + 2 = 0$  
   c) $\sqrt{3} \cot x + 1 = 0$
10. Determine the general solution to the following equations where $x$ is in degree measure. Answer to the nearest degree.

a) $\cos x = -0.639$

b) $\cot x = 0.373$

c) $5 \csc x + 6 = 0$

11. The general solution to the equation $\csc A + 2 = 0$ is

A. $A = \frac{\pi}{6} + n\pi, \ n \in I$

B. $A = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, \ n \in I$

C. $A = \frac{7\pi}{6} + n\pi, \frac{11\pi}{6} + n\pi, \ n \in I$

D. $A = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi, \ n \in I$

12. The general solution to the equation $\sqrt{3} \cot \theta - 1 = 0$ is

A. $\theta = \frac{\pi}{6} + n\pi, \ n \in I$

B. $\theta = \frac{\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi, \ n \in I$

C. $\theta = \frac{\pi}{3} + n\pi, \ n \in I$

D. $\theta = \frac{\pi}{3} + 2n\pi, \frac{4\pi}{3} + 2n\pi, \ n \in I$

13. The smallest positive solution to the equation $\sec x - 5 = 0$, correct to the nearest tenth of a radian, is $x = \underline{____}$
316 Trig - Equations, Identities, ... Lesson #1: Solving First Degree Trigonometric Equations

Answer Key

1. a) Find the $x$-coordinates of the points of intersection of the two graphs
   
   b) $x = \frac{\pi}{3}, \frac{5\pi}{3}$
   
   c) $x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in I$

2. a) Find the $x$-intercepts of the graph
   
   b) $x = \frac{\pi}{4}, \frac{5\pi}{4}$
   
   c) $x = \frac{\pi}{4} + n\pi, n \in I$

3. a) $x = \frac{\pi}{3}, \frac{2\pi}{3}$
   
   b) $x = \frac{3\pi}{4}, \frac{7\pi}{4}$
   
   c) $x = \frac{\pi}{3}, \frac{5\pi}{3}$

4. a) $x = \frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi, n \in I$
   
   b) $x = \frac{3\pi}{4} + n\pi, n \in I$
   
   c) $x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in I$

5. a) $x = 0.93, x = 5.36$
   
   b) $x = 2.03, x = 5.18$
   
   c) $x = 0.34, x = 2.80$

6. a) $x = 0.93 + 2n\pi, 5.36 + 2n\pi, n \in I$
   
   b) $x = 2.03 + n\pi, 5.18 + n\pi, n \in I$
   
   c) $x = 0.34 + 2n\pi, 2.80 + 2n\pi, n \in I$

7. a) $x = \frac{4\pi}{3}, \frac{5\pi}{3}$
   
   b) $x = \frac{5\pi}{6}, \frac{11\pi}{6}$
   
   c) $x = \frac{\pi}{3}, \frac{5\pi}{3}$

8. a) $x = \frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in I$
   
   b) $x = \frac{5\pi}{6} + n\pi, n \in I$
   
   c) $x = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in I$

9. a) $x = \frac{\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi, n \in I$
   
   b) $x = \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi, n \in I$
   
   c) $x = \frac{2\pi}{3} + n\pi, n \in I$

10. a) $x = 130^\circ + 360n^\circ, 230^\circ + 360n^\circ, n \in I$
    
    b) $x = 70^\circ + 180n^\circ, n \in I$
    
    c) $x = 236^\circ + 360n^\circ, 304^\circ + 360n^\circ, n \in I$

11. D
    
    12. C
    
    13. 1.4
Trigonometry -
Equations, Identities, and Modelling Lesson #2:
Solving Second Degree Trigonometric Equations

Warm-Up

Introduction

In this lesson we will be solving second degree equations where the power of the trigonometric function is two (eg. \( \sin^2 x - 3 \sin x = 0 \)). We will:

- factor trigonometric expressions algebraically,
- use a graphical approach to determine an approximate solution,
- use factoring to determine solutions on a domain of length \( 2\pi \) radians, and,
- find the general solution over the domain of real numbers.

Note

Trigonometric equations which can be solved by using identities will be covered in lesson 6.

Factoring Trigonometric Expressions

Just as with polynomial expressions, trigonometric expressions can be factored. The ability to factor trigonometric expressions is a useful skill in two areas:

- solving trigonometric equations (in this lesson)
- proving complicated identities (in lesson 6)

In factoring trigonometric expressions we can apply three basic factoring techniques:

- common factor,
- difference of two squares, and
- factoring trinomials of the form \( ax^2 + bx + c, a \neq 0 \).

Class Ex. #1

Factor the following trigonometric expressions:

- a) \( 8 \tan A + 4 \)
- b) \( \sin^2 x - 3 \sin x \)
- c) \( 4 \sin^2 x - 1 \)
- d) \( \csc^2 x - 3 \csc x - 28 \)
- e) \( 2 \cos^2 x + 7 \cos x - 4 \)
Solving a Second Degree Equation Using a Graphical Approach

Consider the equation \( 2 \sin^2 x = 1 - \sin x \).

a) Use a graphical approach to find the solution to the equation where \( 0 \leq x \leq 2\pi \). Give solutions as exact values.

b) State the general solution to the equation.

Solving a Second Degree Equation Using an Algebraic Approach

Consider the equation \( 2 \sin^2 x = 1 - \sin x \).

a) Use an algebraic approach to find the solution to the equation where \( 0 \leq x \leq 2\pi \). Give solutions as exact values.

b) State the general solution to the equation.
In each of the following:

a) Use an algebraic procedure to find the solution to the equation on the given domain.

b) Write the general solution to the equation

i) \(4 \sin^2 A - 1 = 0, \quad 0 \leq A \leq 2\pi\)

ii) \(\tan^2 x + \tan x = 0, \quad 0 \leq x \leq 2\pi\)

iii) \(\csc^2 x - 3 \csc x - 28 = 0, \quad 0^\circ \leq x \leq 360^\circ\)

iv) \(2 \cos^2 \theta + 5 \cos \theta - 3 = 0, \quad 0 \leq \theta \leq \pi\).

Answer to the nearest degree

Complete Assignment Questions #1 - #10
Assignment

1. Factor the following trigonometric expressions:
   
   a) \(4 \sin^2 \theta - \cos^2 \theta\)  
   b) \(\cot^2 x - \cot x\)  
   c) \(\cot^2 \theta - 1\)
   
   d) \(\sec x \sin^2 x - 0.25 \sec x\)  
   e) \(\sec^4 \theta - 1\)  
   f) \(\sin^2 \theta + 3 \sin \theta + 2\)
   
   g) \(4 \cos^2 A - 4 \cos A - 3\)  
   h) \(2 \sin^2 x - 7 \sin x + 6\)

2. Consider the equation \(2 \cos^2 x + 3 \cos x + 1 = 0\).

   a) Use a graphical approach to find the solution to the equation where \(0 \leq x \leq 2\pi\). Give solutions as exact values.

   b) Use an algebraic approach to find the solution to the equation where \(0 \leq x \leq 2\pi\). Give solutions as exact values.

   c) State the general solution to the equation.
3. Consider the equation \( 2 \sin x \cos x = 3 \sin x \).
   
   a) Use a **graphical** approach to find the solution to the equation where \( 0 \leq x \leq 2\pi \). Give solutions as exact values.
   
   b) Use an **algebraic** approach to find the solution to the equation where \( 0 \leq x \leq 2\pi \). Give solutions as exact values.
   
   c) State the general solution to the equation.

4. Algebraically find the solutions to the following trigonometric equations. Give solutions as exact values.
   
   a) \( 2 \sin^2 \theta + \sin \theta = 0 \) where \( 0 \leq \theta \leq 2\pi \)  
   b) \( 2 \sin^2 x - \sin x = 1 \) where \( 0 \leq x \leq 2\pi \)

   c) \( \cot^2 A + \cot A = 0 \) where \( 0 \leq A \leq \pi \)  
   d) \( 2 \cos^2 x = \sqrt{3} \cos x \) where \( 0 \leq x \leq 2\pi \)
5. Algebraically find the general solutions to the following trigonometric equations. Give solutions as exact values.

\( a) \ 2 \csc^2 \theta - 2 = 3 \csc \theta \)  
\( b) \ 3 \sec \theta = 2 + \sec^2 \theta \)

6. Algebraically find the solutions to the following trigonometric equations, where \( 0 \leq x \leq 2\pi \). Give solutions in decimal form, correct the nearest hundredth.

\( a) \ 6 \sin^2 x - 5 \sin x = -1 \)  
\( b) \ 2 \sin x \cos x + 2 \sin x = \cos x + 1 \)
7. The diagram below shows the graphs of the functions \( y = 6 \sin^2 x \) and \( y = 6 \sin x - 1 \) where \( 0 \leq x \leq 2\pi \).

![Graph of functions](image)

a) **Explain how** you could use this diagram to estimate the solution to the equation \( 6 \sin^2 x - 6 \sin x + 1 = 0 \), where \( 0 \leq x \leq 2\pi \).

b) Use an algebraic approach to find the solutions to the equation

\[
6 \sin^2 x - 6 \sin x + 1 = 0,
\]

where \( 0 \leq x \leq 2\pi \).

Give the solution correct to the nearest hundredth.

c) **Explain how** you could use this diagram to estimate the solution to the equation \( 6 \sin^2 x (6 \sin x - 1) = 0 \), where \( 0 \leq x \leq 2\pi \).

d) Use an algebraic approach to find the solutions to the equation \( 6 \sin^2 x (6 \sin x - 1) = 0 \), where \( 0 \leq x \leq 2\pi \). Give the solution correct to the nearest hundredth.
8. Which solutions are correct for the equation $12 \sin^2 x - 11 \sin x + 2 = 0$?

A. $\sin x = 3, 8$
B. $\sin x = \frac{11}{12}, -2$
C. $\sin x = \frac{2}{3}, \frac{1}{4}$
D. $\sin x = \frac{2}{3}, -\frac{1}{4}$

9. The number of solutions of the equation $2 \cos^2 x + \cos x - 1 = 0$, where $-8\pi \leq x \leq 8\pi$ is _____.

10. If angle $A$ is acute and $\log_4 (\sin^2 A) = -1$, then the value of $A$, to the nearest tenth of a radian, is _____.

---

**Answer Key**

1. a) $(2 \sin \theta - \cos \theta)(2 \sin \theta + \cos \theta)$
   b) $\cot x(\cot x - 1)$
   c) $(\cot x - 1)(\cot x + 1)$
   d) $\sec x(\sin x + 0.5)(\sin x - 0.5)$
   e) $(\sec x - 1)(\sec x + 1)(\sec^2 x + 1)$
   f) $(\sin x + 2)(\sin x + 1)$
   g) $(2 \cos x - 3)(2 \cos A + 1)$
   h) $(\sin x - 2)(2 \sin x - 3)$

2. a) $\frac{2\pi}{3}, \frac{4\pi}{3}$
   b) $\frac{2\pi}{3}, \frac{4\pi}{3}$
   c) $x = \frac{2\pi}{3} + 2n\pi, \pi + 2n\pi, \frac{4\pi}{3} + 2n\pi, n \in \mathbb{I}$

3. a) $0, \pi, 2\pi$
   b) $0, \pi, 2\pi$
   c) $x = n\pi, n \in \mathbb{I}$

4. a) $0, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$
   b) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
   c) $\frac{\pi}{2}, \frac{3\pi}{4}$
   d) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{11\pi}{6}$

5. a) $x = \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi, n \in \mathbb{I}$
   b) $x = 2n\pi, \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in \mathbb{I}$

6. a) $0.34, 0.52, 2.62, 2.80$
   b) $0.52, 2.62, 3.14$

7. a) Find the x-coordinates of the points of intersection of the two graphs
   b) $0.21, 0.91, 2.23, 2.93$
   c) Find the x-intercepts of each graph
   d) $0.00, 0.17, 2.97, 3.14, 6.28$

8. C
9. 24
10. 0.5
Trigonometry -
Equations, Identities, and Modelling Lesson #3:
Equations which require a Graphical Solution

Equations which require a Graphical Solution

Some equations cannot be solved by an algebraic approach. In these cases we use a graphical approach to estimate the solution.

Warm-Up

Consider the equation $3 \sin x = x$.

a) Use the following graphical method to estimate the solution to the equation on the domain $0 \leq x \leq 2\pi$.

- Choose a window with appropriate values which will enable us to find the points of intersection of the graphs of $y = 3 \sin x$ and $y = x$.
  
  Use window $x: [0, 2\pi, \frac{\pi}{6}]$ $y: [ ]$

- Graph $Y_1 = x$

- Graph $Y_2 = 3 \sin x$

- Use INTERSECT to determine, to the nearest tenth, the $x$-coordinate(s) of the point(s) of intersection.

b) To determine the general solution to the equation $3 \sin x = x$ we need to recognize that the diagonal line ($y = x$) and the periodic function ($y = 3 \sin x$) will intersect once more to the left of the origin and nowhere else on the domain of real numbers.

- Write the general solution to the equation $3 \sin x = x$.

c) Describe how we could find approximate solutions to the equation $3 \sin x = x$ using the ZERO feature of the calculator.
Use a graphical approach and the ZERO feature of the calculator to find the solution to the equation \( x^3 - x^2 = 2 \sin x \). Give the answers to the nearest tenth.

**Assignment**

1. Use a graphical approach to solve the following equations to the nearest hundredth where necessary.
   a) \( \cos x = x + 1 \)   
   b) \( x - 2\sin x = 0 \)
   c) \( \sin 2x = x \),
   d) \( \cos x = x^2 \)

2. a) Graph \( y = 4 \sin x - x \) on a domain \(-2\pi \leq x \leq 2\pi\). Show the graph on the grid.
   b) Solve the equation \( x = 4 \sin x \), where \( x \in \mathbb{R} \), to the nearest hundredth.
   c) Explain how you could use the graph to solve the equation \( x = 4 \sin x + 3 \), \(-2\pi \leq x \leq 2\pi\).
   d) Solve the equation \( x = 4 \sin x + 3 \) where \( x \in \mathbb{R} \), to the nearest hundredth.
3. a) Graph \( y = x^2 + \sin 6x \) on a domain \( -\pi \leq x \leq \pi \).
    Show the graph on the grid.

b) Solve the equation \( x^2 = -\sin 6x \) where \( x \in \mathbb{R} \),
    to the nearest hundredth.

c) Explain how you could use the graph to solve
    the equation \( x^2 = -\sin 6x + 1, -\pi \leq x \leq \pi \).

d) Solve the equation \( x^2 = -\sin 6x + 1 \) where \( x \in \mathbb{R} \), to the nearest hundredth.

4. How many solutions are there to the equation \( \cos^3 x = \sin^3 x + 0.5 \) where \( -\pi \leq x \leq \pi \)?
   
   A. 1
   B. 2
   C. 3
   D. more than 3

5. How many solutions are there to the equation \( \cos 3x + \frac{1}{2}x = 0 \) ?
   
   A. 2
   B. 3
   C. 4
   D. more than 4

6. The smallest positive solution to the equation \( \sec x - x = 0 \), correct to
    the nearest tenth of a radian, is \( x = \) _____ .

7. The number of solutions to the equation \( x \sin 2x = 0 \) where \( -2\pi \leq x \leq 2\pi \) is _____.
Answer Key

1. a) 0  b) –1.90, 0, 1.90  c) –0.95, 0, 0.95  d) –0.82, 0.82

2. b) –2.47, 0.00, 2.47  
   c) Find the x coordinates of the points of intersection of the graph with the line y = –3  
   d) 3.11

3. b) –0.48, 0.00, 0.58, 0.89  
   c) Find the x coordinates of the points of intersection of the graph with the line y = 1  
   d) –1.38, –1.07, –0.63, 0.21, 0.34, 1.04

4. B  
5. B  
6. 4.9  
7. 9
Trigonometry -
Equations, Identities, and Modelling Lesson #4:
Solving Equations Involving Multiple Angles

Warm-Up #1
In this lesson we will be solving equations involving multiple angles, eg. \( \sin 3x = 1 \).
We will:
• introduce a graphical approach to determine an approximate solution to first degree trigonometric equations involving multiple angles
• find the general solution over the domain of real numbers
• algebraically find the solutions to an equation involving multiple angles.

Warm-Up #2
Graphically Exploring Solutions to Multiple Angle Equations
Consider the equations \( \cos x = -\frac{\sqrt{3}}{2} \) and \( \cos 2x = -\frac{\sqrt{3}}{2} \) where \( x \) is in degree measure.

a) Use a graphical approach to determine the solution to the equation
\( \cos x = -\frac{\sqrt{3}}{2} \) where \( 0 \leq x \leq 360^\circ \).

b) Use a graphical approach to determine the solution to the equation
\( \cos 2x = -\frac{\sqrt{3}}{2} \) where \( 0 \leq x \leq 360^\circ \).

c) Compare the solutions to the two equations in terms of:
• the number of solutions
• the values of \( x \).

d) Complete the following
i) The general solution to \( \cos x = -\frac{\sqrt{3}}{2} \) is ________________________________

ii) The general solution to \( \cos x = -\frac{\sqrt{3}}{2} \) consists of two sets of answers which differ by _____ degrees because the graph of \( y = \cos x \) has a period of _____ degrees.

iii) The general solution to \( \cos 2x = -\frac{\sqrt{3}}{2} \) will consist of two sets of answers which differ by _____ degrees because the graph of \( y = \cos 2x \) has a period of _____ degrees.

iv) The general solution to \( \cos 2x = -\frac{\sqrt{3}}{2} \) is ________________________________.
Solving a Multiple Angle Equation Using a Graphical Approach

a) Given \( \tan 2x = \sqrt{3} \), where \( 0 \leq x \leq 2\pi \), find the exact values of \( x \) using a graphical approach.

b) State the general solution to the equation \( \tan 2x = \sqrt{3} \).

c) Complete the following statement:

The general solution consists of answers which differ by _____ radians because the graph of \( y = \tan 2x \) has a period of _____ radians.

Warm-Up #3  Algebraically Exploring Solutions to Multiple Angle Equations

Consider the equation \( \sin 3x = \frac{\sqrt{2}}{2} \).

a) Complete the following to solve the equation \( \sin 3x = \frac{\sqrt{2}}{2} \), where \( 0 \leq x \leq 2\pi \)

\[
\sin 3x = \frac{\sqrt{2}}{2} \\
\text{Quadrants} \quad \text{and} \\
\text{Reference angle} =
\]

\[
3x = \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\
3x = \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\
x = \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\ \quad \text{or} \\ \quad \text{or}
\]

b) State the general solution to the equation \( \sin 3x = \frac{\sqrt{2}}{2} \).

c) Verify the solution using a graphical approach.
• The general solution consists of two sets of answers which differ by $\frac{2\pi}{3}$ radians because the graph of $y = \sin 3x$ has a period of $\frac{2\pi}{3}$ radians.

**Solving a Multiple Angle Equation Using an Algebraic Approach**

Use the following procedure to solve multiple angle equations:

1. Find the domain for the multiple angle.
2. Solve for the multiple angle between 0 and $2\pi$ using the CAST rule and reference angle.
3. Add the period of the trigonometric graph of the multiple angle to each of the answers in step 2 until you cover the domain in step 1.

**Class Ex. #2**

Consider the equation $\cos 2x = -\frac{1}{2}$.

a) Find the exact values of $x$ using an algebraic approach where $0 \leq x \leq 2\pi$.

b) State the general solution to the equation $\cos 2x = -\frac{1}{2}$.

c) Complete the following statement.

The general solution consists of answers which differ by _____ radians because the graph of $y = \cos 2x$ has a period of _____ radians.

d) Verify the solution using a graphical approach.
Assignment

1. a) Given $\sin 2x = \frac{\sqrt{3}}{2}$, where $0 \leq x \leq 2\pi$, find the exact values of $x$ using a graphical approach.

   b) Find the general solution to $\sin 2x = \frac{\sqrt{3}}{2}$.

2. a) Given $\cot 2x = 1$, where $0 \leq x \leq 2\pi$, find the exact values of $x$ using a graphical approach.

   b) Solve $\cot 2x = 1$ where $x \in \mathbb{R}$.

3. Find the general solution to $\cos \frac{1}{2}x = \frac{\sqrt{3}}{2}$ using a graphical approach.

4. a) Use an algebraic approach to solve the equation $\sin 2x = \frac{\sqrt{2}}{2}$, $0 \leq x \leq 2\pi$

   b) State the general solution to the equation $\sin 2x = \frac{\sqrt{2}}{2}$

5. a) Use an algebraic approach to solve the equation $\sec 3x = -2$, $0 \leq x \leq 2\pi$

   b) State the general solution to the equation $\sec 3x = -2$
6. Use an algebraic approach to determine the general solution to the equation \( \csc 2x = \frac{2\sqrt{3}}{3} \).

7. The graph of \( y = 2 \cos 3x + 1 \), is displayed on a graphic calculator.

   a) Describe the effects of the parameters 2, 3 and 1 on the graph of \( y = \cos x \).

   b) A student was asked to find all the values of \( \theta \) which satisfy the equation \( \cos 3x = -\frac{1}{2}, \ 0 \leq x \leq \pi \).

   Explain how the student can find these values from the graph above and mark these points on the grid.

   c) Show how to find these values by solving algebraically \( \cos 3x = -\frac{1}{2}, \ 0 \leq x \leq \pi \).
8. a) Factor the expression \(2 \sin^2 x - \sin x - 1\) and hence solve the equation \(2 \sin^2 x - \sin x - 1 = 0\) for \(0 \leq x \leq 2\pi\).

b) Describe how the solution of \(2 \sin^2 \left(\frac{1}{2}x\right) - \sin \left(\frac{1}{2}x\right) - 1 = 0\), \(0 \leq x \leq 2\pi\) relates to the solution of \(2 \sin^2 x - \sin x - 1 = 0\), \(0 \leq \theta \leq 2\pi\). Find these solutions.

9. Which of the following is NOT a solution to the equation \(2 \sin 3x = 0\)?

- A. \(\frac{\pi}{3}\)
- B. \(\frac{\pi}{2}\)
- C. \(\frac{4\pi}{3}\)
- D. \(2\pi\)

10. If \(p\) and \(q\) are two solutions to the equation \(\tan 5x = 0.8\pi\), which of the following statements CANNOT be true?

- A. \(p - q = 0.8\pi\)
- B. \(p - q = \pi\)
- C. \(p - q = 2.5\pi\)
- D. \(p - q = 5\pi\)

11. The smallest positive solution to the equation \(\sin 4x = 0.48\), correct to the nearest hundredth of a radian, is \(x = _____\).
In mathematics it is important to understand the difference between an equation and an identity.

\(2x^2 + 3 = 11\) is an equation. It is only true for certain values of the variable \(x\). The solutions to this equation are \(-2\) and \(2\) which can be verified by substituting these values into the equation.

\((x + 1)^2 = x^2 + 2x + 1\) is an identity. It is true for all values of the variable \(x\).

**Warm-Up  Reviewing Identities**

Recall the basic trigonometric identities:

<table>
<thead>
<tr>
<th>Basic Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta = \frac{y}{r} )</td>
</tr>
</tbody>
</table>

where \( x^2 + y^2 = r^2 \)

We have also met the reciprocal trigonometric identities:

<table>
<thead>
<tr>
<th>Reciprocal Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sec \theta = \frac{1}{\cos \theta} )</td>
</tr>
</tbody>
</table>

These identities are on the formula sheet.

We can use the Basic and Reciprocal trigonometric identities to prove the Quotient and Pythagorean identities.

Before doing this we will verify some identities using a particular case.
Verifying Identities for a Particular Case

When verifying an identity we must treat the left side (LS) and the right side (RS) separately and work until both sides represent the same value.

This technique does not prove that an identity is true for all values of the variable - only for the value of the variable being verified.

Verify the following identities for the value given.

a) \(\tan x = \frac{\sin x}{\cos x}\) for \(x = 60^\circ\)

b) \(\tan^2 x + 1 = \sec^2 x\) for \(x = \frac{\pi}{6}\)

Proving Identities

We can use identities to derive other identities. When proving an identity we must:

• treat the left side (LS) and the right side (RS) separately

• work until both sides represent the same expression.

Remember:

• Do not make the mistake of assuming the answer by writing the LS = RS at the start of a proof and do not move terms from one side to the other.

Use the basic identities to prove the identity \(\tan \theta = \frac{\sin \theta}{\cos \theta}\), where \(\cos \theta \neq 0\). Explain why there is the restriction \(\cos \theta \neq 0\).
Use the basic identities to prove the identity $1 + \tan^2 A = \sec^2 A$.

In the same way the basic identities can be used to prove the following:

**Quotient Identities**

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}
\]

*These identities are on the formula sheet.*

and

**Pythagorean Identities**

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta
\]

*These identities are on the formula sheet.*

- These identities can be written in several ways and this should be remembered in trying to prove more difficult identities in the next lesson. For example

  \[
  \sin^2 \theta = 1 - \cos^2 \theta \quad \cos^2 \theta = 1 - \sin^2 \theta
  \]

  \[
  \tan^2 \theta = \sec^2 \theta - 1 \quad \cot^2 \theta = \csc^2 \theta - 1
  \]

- We use the basic trigonometric identities in terms of $x, y$ and $r$ to prove *only* the Quotient and Pythagorean Identities.

- You will be asked to verify the remaining Quotient and Pythagorean Identities in the Assignment.

- Before considering more complex identities in the next lesson we need to review some skills in simplification and factoring which will help in the proofs.
Simplifying Trigonometric Expressions

Class Ex. #4
Express each as a single trigonometric ratio. Use a graphing calculator to verify.

a) \( \frac{\sin^2 x}{\cos^2 x} + 1 \)  
b) \( \sin x + \cot x \cos x \)

Class Ex. #5
Express \( \frac{2 \tan A}{1 + \tan^2 A} \) in terms of \( \sin A \) and \( \cos A \) and write in simplest form.

Class Ex. #6
Factor the following trigonometric expressions

a) \( 3 \cos^4 \theta - 3 \sin^4 \theta \)  
b) \( \sin^2 \theta + \sin^2 \theta \cot^2 \theta \)

Complete Assignment Questions #1 - #15
Assignment

1. Verify the following identities for the given value of the variable:
   a) \( \cot \theta = \frac{\cos \theta}{\sin \theta} \) for \( \theta = \frac{\pi}{3} \)  
   b) \( \sin^2 \theta + \cos^2 \theta = 1 \) for \( \theta = \frac{\pi}{4} \)

2. Verify the identity \( 1 + \cot^2 x = \csc^2 x \) for the given values:
   a) \( x = \frac{\pi}{6} \)  
   b) \( x = \frac{4\pi}{3} \)

3. Use the identities discussed in this lesson to state whether the following are true or false. (Written work is not required.)
   a) \( \cos^2 \theta = 1 + \sin^2 \theta \)  
   b) \( (\sin x)(\csc x) = 1 \)  
   c) \( \sin \theta = \pm \sqrt{1 - \cos^2 \theta} \)
   
   d) \( (\tan x)(\cot x) = 1 \)  
   e) \( \tan^2 \theta - \sec^2 \theta = 1 \)
4. Write each expression as a single trigonometric ratio or as the number 1.
   a) \( \sin^2 x - 1 \)  
   b) \( \frac{\cos t}{\sin t} \)  
   c) \( \frac{1}{\sec \theta} \)
   
   d) \( (\sec t)(\sin^2 t)(\csc t) \)  
   e) \( \csc^2 x - \cot^2 x \)  
   f) \( \sin \theta + (\cot \theta)(\cos \theta) \)

5. Factor to write each in a simpler form.
   a) \( \sec x \sin^2 x - \sec x \)  
   b) \( \sin^4 \theta - \cos^4 \theta \)

6. Very often in proving identities it is simpler to try to express each side in terms of only \( \sin x, \cos x, \) or both. Express each of the following in terms of only \( \sin x, \cos x, \) or both.
   a) \( \tan^2 x \)  
   b) \( \frac{\tan x}{\sin x} \)  
   c) \( \frac{\tan x}{\csc x} \)  
   
   d) \( \frac{1}{1 + \cot^2 x} \)  
   e) \( \csc x - \sin x \)  
   f) \( 1 - \csc^2 x \)  
   
   g) \( \frac{1 + \cot^2 x}{\sec^2 x} \)  
   h) \( \frac{\cos^2 x - 1}{\tan x} \)  
   i) \( \frac{1 + \cot^2 x}{\cot^2 x} \)
In questions #7 - #11 assume the appropriate restrictions.

7. \( \frac{\cos \theta}{1 - \sin^2 \theta} \) is equal to
   A. \( \sec \theta \)
   B. \( \csc \theta \)
   C. \( \sin \theta \)
   D. \( \tan \theta \)

8. \( \frac{\tan^2 x + 1}{\sec x} \) is equal to
   A. \( \sec x \)
   B. \( \csc x \)
   C. \( \sin x \)
   D. \( \tan x \)

9. \( \frac{\csc \theta}{\cot \theta} \) is equal to
   A. \( \cos \theta \)
   B. \( \sin \theta \)
   C. \( \sec \theta \)
   D. \( \tan \theta \)

10. The expression \( \frac{\tan A \cos^2 A}{\sec A} \), expressed in terms of \( \sin A \) is
    A. \( \frac{\sin A}{1 - \sin^2 A} \)
    B. \( \frac{1 - \sin^2 A}{\sin A} \)
    C. \( \sin^2 A \)
    D. \( \sin A - \sin^3 A \)

11. \( \sec \theta - \cos \theta \) is equal to
    A. \( \frac{1 - \cos \theta}{\cos \theta} \)
    B. \( \sin^2 \theta \)
    C. \( \frac{1 - 2 \cos \theta}{\cos \theta} \)
    D. \( \sin \theta \tan \theta \)

12. Which of the following is NOT an identity?
    A. \( \cos^2 x + \sin^2 x = 1 \)
    B. \( \sin x + \cos x = 1 \)
    C. \( \sec^2 x - \tan^2 x = 1 \)
    D. \( \tan x \cot x = 1 \)
13. If \( \tan x \neq 0 \), \( \cos x \neq 0 \), \( \cot x \neq 0 \) then 
\[
\frac{1}{\tan x \cos x \cot x}
\]
is equal to 

A. \( \frac{1}{\sin x} \)  
B. \( \sin x \)  
C. \( \frac{1}{\cos x} \)  
D. \( \cos x \)

14. If \( \sin x \neq 0 \), \( \cos x \neq 0 \) then 
\[
\frac{\tan x \cos x}{3 \sec x \cot x}
\]
is equal to 

A. \( \frac{1}{3} \)  
B. 3  
C. \( \frac{1}{3} \sin^2 x \)  
D. \( \frac{1}{3} \csc^2 x \)

15. When verifying the identity \( \cot^2 \theta + 1 = \csc^2 \theta \) for \( \theta = \frac{\pi}{7} \), the value on each side of the identity, to the nearest tenth, is _____.

**Answer Key**

3. b), c) and d) are true.
4. a) \(- \cos^2 x\)  b) \(\cot t\)  c) \(\cos \theta\)  d) \(\tan t\)  e) 1  f) \(\csc \theta\)
5. a) \(\frac{\sin^2 x}{\cos^2 x}\)  b) \(\frac{1}{\cos x}\)  c) \(\frac{\sin^2 x}{\cos x}\)  d) \(\frac{\sin^2 x}{\cos x}\)  e) \(\frac{\cos^2 x}{\sin x}\)
6. f) \(- \frac{\cos^2 x}{\sin^2 x}\)  g) \(\frac{\cos^2 x}{\sin^2 x}\)  h) \(- \sin x \cos x\)  i) \(\frac{1}{\cos^2 x}\)
Trigonometry -
Equations, Identities, and Modelling Lesson #6:
Trigonometry Identities Part 2

**Warm-Up**

In this lesson we will verify and prove more complex trigonometric identities using the skills we learned from the previous lesson. Some useful steps or hints when trying to prove a trigonometric identity are listed below.

**Hints in Proving an Identity**

1. Begin with the more complex side.
2. If possible, use known identities given on the formula sheet, e.g. try to use the Pythagorean identities when squares of trigonometric functions are involved.
3. If necessary change all trigonometric ratios to sines and/or cosines, e.g. replace \( \tan \theta \) by \( \frac{\sin \theta}{\cos \theta} \), or \( \sec \theta \) by \( \frac{1}{\cos \theta} \).
4. Look for factoring as a step in trying to prove an identity.
5. If there is a sum or difference of fractions, write as a single fraction.
6. Occasionally, you may need to multiply the numerator or denominator of a fraction by its conjugate.

**Class Ex. #1**

Consider the statement \( \frac{1}{\cos \theta} - \cos \theta = \sin \theta \tan \theta \).

a) Verify the statement is true for \( \theta = \frac{\pi}{3} \).

<table>
<thead>
<tr>
<th>L.S.</th>
<th>R.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Use a graphing calculator to show that the statement is probably an identity.

d) State the restrictions in terms of \( \theta \).

Note that the restrictions can be verified from the graph.
Consider the statement \( \frac{\sin \theta}{\tan \theta} = \cos \theta \)

a) Verify the statement is true for \( \theta = \frac{\pi}{6} \)

<table>
<thead>
<tr>
<th>L.S.</th>
<th>R.S.</th>
</tr>
</thead>
</table>

b) Use a graphing calculator to show that the statement is probably an identity.

c) Prove the statement is an identity using an algebraic approach

<table>
<thead>
<tr>
<th>L.S.</th>
<th>R.S.</th>
</tr>
</thead>
</table>

d) State the restrictions in terms of \( \theta \).

---

Prove the following identity algebraically.

\[
\sin x \cos^2 x + \sin^3 x = \frac{1}{\csc x}
\]

| L.S. | R.S. |
Class Ex. #4

Prove that \( \frac{\sin A + \tan A}{1 + \cos A} = \frac{1}{\cot A} \) is an identity.

For what values of \( A \) is this identity undefined?

Class Ex. #5

Prove the identity \( \frac{\sec^2 \theta}{\sec^2 \theta - 1} = \csc^2 \theta \).

Using Identities to Solve Equations

Solve the following equations where \( 0 \leq x \leq 2\pi \).

a) \( 2 \cos^2 x - 3 \sin x = 0 \)  

b) \( \sin x - \sqrt{3} \cos x = 0 \)

Complete Questions #1 - #11
1. Consider the statement \( \frac{\cos \theta - \sin \theta}{\cos \theta} = \sin^2 \theta - \tan \theta + \cos^2 \theta \).

   a) Verify the statement is true for \( \theta = \frac{\pi}{4} \).
   b) Prove the statement is an identity.

   c) State, and give reasons, for any restrictions.

2. Consider the statement \( \frac{\cot \theta - 1}{\tan \theta - 1} = -\frac{1}{\tan \theta} \).

   a) Verify the statement is true for \( \theta = \frac{\pi}{3} \).
   b) Prove the statement is an identity.

   c) State, and give reasons, for any restrictions.
3. In each of the following:

i) verify the possibility of an identity using a graphing calculator
ii) prove the identity using an algebraic approach
iii) state any restrictions.

\[ \frac{\tan \theta \cos \theta}{\sin \theta} = 1 \]

a) \[ \sec^2 x - \sin^2 x = \cos^2 x + \tan^2 x \]

b) \[ \sec^2 x - \sin^2 x = \cos^2 x + \tan^2 x \]

4. Prove the following identities using an algebraic approach.

a) \( (1 - \cos^2 x)(\csc x) = \sin x \)

b) \( (\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta \)

c) \[ \frac{1 - \cos x}{\sin x} = \frac{\tan x - \sin x}{\tan x \sin x} \]

d) \[ \frac{2}{1 - \sin \theta} + \frac{2}{1 + \sin \theta} = 4 \sec^2 \theta \]
e) \( \frac{1 + \cos x}{\tan x + \sin x} = \cot x \)

f) \( \sec x - \cos x = \frac{\sin x}{\cot x} \)

5. Use conjugates to prove the following identities using an algebraic approach.

a) \( \frac{1}{1 - \sin A} = \frac{1 + \sin A}{\cos^2 A} \)

b) \( \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A} \)
6. Prove that \[
\frac{2 \cos x - 1}{2 \cos^2 x - 7 \cos x + 3} = \frac{1}{\cos x - 3}
\]
is an identity.

For what values of \(x\) is this identity undefined?

7. Solve for \(x\) as an exact value where \(0 \leq x \leq 2\pi\).

- a) \(3 - 3 \sin x - 2 \cos^2 x = 0\)
- b) \(\tan^2 x - 1 = \sec x\)
- c) \(\sin^2 x - \cos^2 x = 0\)
8. Solve, to the nearest tenth, the equations $7 \sec^2 x + 2 \tan x - 6 = 2 \sec^2 x + 2$, $0 \leq x \leq 2\pi$.

9. The identity $\frac{\sec x + 1}{\sec x - 1} + \frac{\cos x + 1}{\cos x - 1} = 0$ has restrictions

A. $x \neq 2n\pi, \frac{\pi}{2} + 2n\pi, n \in \mathbb{I}$
B. $x \neq 2n\pi, \frac{\pi}{2} + n\pi, n \in \mathbb{I}$
C. $x \neq \pi + 2n\pi, \frac{\pi}{2} + 2n\pi, n \in \mathbb{I}$
D. $x \neq 2n\pi, n \in \mathbb{I}$

10. The value of $n$ to the nearest tenth for which the statement below is an identity, is ______.

$$\frac{1 + \sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} = \frac{n}{\cos \theta}$$

11. If $\frac{p}{2} \cos^2 \frac{\pi}{5} + \frac{p}{2} \sin^2 \frac{\pi}{5} = 4$, the value of $p$, to the nearest tenth, is ______.

**Answer Key**

1. a) both sides equal 0  c) $\theta \neq \frac{\pi}{2} + n\pi$, $n \in \mathbb{I}$

2. a) both sides equal $-\frac{\sqrt{3}}{3}$  c) $\theta \neq n\pi, \frac{\pi}{4} + n\pi, \frac{\pi}{2} + n\pi, n \in \mathbb{I}$ or $\theta \neq \frac{\pi}{4} + n\pi, n \frac{\pi}{2}, n \in \mathbb{I}$

3. a) $x \neq n\pi, \frac{\pi}{2} + n\pi, n \in \mathbb{I}$ or $\theta \neq n\pi, \frac{\pi}{2}, n \in \mathbb{I}$ b) $x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{I}$

6. $x \neq \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi, n \in \mathbb{I}$

7. a) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$  b) $\frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{3}$  c) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

8. 0.5, 2.4, 3.7, 5.5  9. B  10. 2.0  11. 8.0
Consider the statement $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$.

a) Determine whether or not the statement can be verified using $\alpha = 60^\circ$ and $\beta = 30^\circ$.

b) What can we say about the statement $\sin(\alpha + \beta) = \sin \alpha + \sin \beta$?

Use exact values to verify the following statements:

a) $\sin(60 + 30)^\circ = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

b) $\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \sin \frac{\pi}{3} \cos \frac{\pi}{6} - \cos \frac{\pi}{3} \sin \frac{\pi}{6}$

c) $\cos(30^\circ + 60^\circ) = \cos 30^\circ \cos 60^\circ - \sin 30^\circ \sin 60^\circ$

d) $\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos \frac{\pi}{2} \cos \frac{\pi}{4} + \sin \frac{\pi}{2} \sin \frac{\pi}{4}$
**Addition Identities**

Warm-Up #2 is an example of verifying the addition identities, sometimes called the sum and difference identities.

\[
\begin{align*}
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta
\end{align*}
\]

*These identities are on the formula sheet.*

**Class Ex. #1**

Use a difference identity to find the exact value of \( \sin 15^\circ \).

**Class Ex. #2**

Find the exact value of \( \sec \frac{5\pi}{12} \).

**Class Ex. #3**

Simplify the following:

a) \( \sin 100^\circ \cos 10^\circ - \cos 100^\circ \sin 10^\circ \)

b) \( \cos \left( \frac{1}{4} \pi - \theta \right) \cos \left( \frac{1}{4} \pi + \theta \right) - \sin \left( \frac{1}{4} \pi - \theta \right) \sin \left( \frac{1}{4} \pi + \theta \right) \)
Given \( \cos \alpha = \frac{3}{5} \) and \( \cos \beta = \frac{5}{13} \), where \( 0 \leq \alpha \leq \frac{\pi}{2} \) and \( \frac{3\pi}{2} \leq \beta \leq 2\pi \), find the exact value of \( \cos(\alpha + \beta) \).

Consider the function \( f(x) = \sin \left( \frac{\pi}{4} + x \right) - \sin \left( \frac{\pi}{4} - x \right) \).

a) Simplify \( f(x) \).

b) Use the result in a) to solve the equation \( f(x) = -1 \) where \( 0 \leq x \leq 2\pi \).

c) Verify the solutions in b) graphically.

Complete Assignment Questions #1 - #12
Assignment

1. Simplify using the addition identities.
   a) \( \cos(180 - B)^\circ \)  
   b) \( \sin\left(\frac{\pi}{2} - x\right) \)
   c) \( \cos(90 + r)^\circ \)  
   d) \( \sin(\pi + x) \)

2. Simplify and evaluate the following:
   a) \( \sin 70^\circ \cos 20^\circ + \cos 70^\circ \sin 20^\circ \)  
   b) \( \cos 170^\circ \cos 50^\circ + \sin 170^\circ \sin 50^\circ \)
   c) \( \sin \frac{\pi}{3} \cos \frac{\pi}{6} - \sin \frac{\pi}{6} \cos \frac{\pi}{3} \)  
   d) \( \sin^2\left(\frac{\pi}{2} - x\right) + \sin^2 x \)

3. Use exact values to show that:
   a) \( \sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4} \)  
   b) \( \cos \frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4} \)
4. Find the exact value of each of the following:

a) \( \cos \frac{13\pi}{12} \)  

b) \( \csc 105^\circ \)

5. Given \( \sin \alpha = \frac{3}{5} \) and \( \sin \beta = \frac{7}{25} \), and \( \alpha \) and \( \beta \) are both acute angles, show that \( \cos (\alpha + \beta) = \frac{3}{5} \).

6. Given that \( \tan X = \frac{12}{5} \) and \( \tan Y = \frac{4}{3} \), where \( 0 \leq X \leq \frac{\pi}{2} \) and \( \pi \leq Y \leq \frac{3\pi}{2} \), find the exact values for \( \cos(X + Y) \) and \( \sin(X + Y) \).
7. Consider the function \( f(x) = \cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right) \).

   a) Simplify \( f(x) \).

   b) Use the result in a) to solve the equation \( f(x) = 1 \) where \( 0 \leq x \leq 2\pi \).

   c) Verify the solution(s) in b) graphically.

8. Prove the following identities:

   a) \( \frac{\sin(A - B)}{\cos A \cos B} = \tan A - \tan B \)

   b) \( (\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 2[1 + \cos(A - B)] \)
c) \( \sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y \)

d) \( \cos(x + y) \cos(x - y) = \cos^2 x + \cos^2 y - 1 \)

**Multiple Choice** 9. If \( \cos(\alpha + \beta) = 0.8320 \) and \( \cos(\alpha - \beta) = 0.4358 \) then the value of \( \cos \alpha \cos \beta \) is

A. 1.2678  
B. 0.6339  
C. 0.3962  
D. 0.1981
10. The value of \( \cos(\pi + y) - \cos(\pi - y) \) is
   A. 0
   B. 2
   C. -2
   D. Dependent on the value of \( y \).

11. Given \( \csc x = -\frac{17}{15} \) where \( \frac{3\pi}{2} \leq x \leq 2\pi \) and \( \cot y = -\frac{3}{4} \) where \( \frac{\pi}{2} \leq y \leq \pi \), the value of \( \cos(x - y) \) is
   A. \( \frac{84}{85} \)
   B. \( \frac{36}{35} \)
   C. \( \frac{84}{85} \)
   D. \( \frac{36}{35} \)

12. If \( \sin(A + B) = 0.75 \) and \( \sin(A - B) = 0.43 \), then the value of \( \sin B \cos A \), to the nearest hundredth, is _____.
Consider the statement \( \sin 2\theta = 2 \sin \theta \).

a) Determine whether or not the statement can be verified using \( \theta = \frac{\pi}{6} \).

b) What can we say about the statement \( \sin 2\theta = 2 \sin \theta \)?

Use exact values to verify the following statements:

a) \( \sin \frac{\pi}{3} = 2 \sin \frac{\pi}{6} \cos \frac{\pi}{6} \)

b) \( \cos \frac{\pi}{2} = \cos^2 \frac{\pi}{4} - \sin^2 \frac{\pi}{4} \)

Warm-Up #2 is an example of verifying the double angle identities. The double angle identities for sine and cosine are shown.

\[
\begin{align*}
\sin 2\theta &= 2 \sin \theta \cos \theta \\
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta
\end{align*}
\]

These identities are on the formula sheet.

The identity for \( \cos 2\theta \) can also be written in the following forms, which are given on the formula sheet. You will be asked to prove all three forms of the identity in the assignment.

\[
\begin{align*}
\cos 2\theta &= 2 \cos^2 \theta - 1 \\
\cos 2\theta &= 1 - 2 \sin^2 \theta
\end{align*}
\]

These identities are on the formula sheet.
Use an addition identity to prove the double angle identity \( \sin 2A = 2 \sin A \cos A \).

Consider the identity \( \frac{2 \tan x}{1 + \tan^2 x} = \sin 2x \).

a) Describe how to use a graphing calculator to verify the identity.

b) Prove the identity.

Express each of the following in terms of a single trigonometric function.

a) \( 2 \sin 4x \cos 4x \)  

b) \( \cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A \)  

c) \( \sin \frac{5}{2}x \cos \frac{5}{2}x \)

**Solving Equations Using Double Angle Identities**

Jenny graphed the following equations on her graphing calculator.

\[ y = -\cos 2x \]
\[ y = 3 \sin^2 x - 2 \]

a) Describe how to use the graphs to solve the equation

\[ 3 \sin^2 x + \cos 2x - 2 = 0 \]

where \( 0 \leq x \leq 2\pi \).

Mark these points with DOTS on the grid.
b) Find the solution to the equation \(3 \sin^2 x + \cos 2x - 2 = 0\), where \(0 \leq x \leq 2\pi\), by algebraic means, as exact values.

c) Describe how to use the graphs to solve the equation
\[-\cos 2x (3 \sin^2 x - 2) = 0, \text{ where } 0 \leq x \leq 2\pi.
\]
Mark these points with a SQUARE on the grid.

d) Solve algebraically the equation \(-\cos 2x (3 \sin^2 x - 2) = 0\), where \(0 \leq x \leq 2\pi\), to the nearest tenth.

Complete Assignment Questions #1 - #8

Assignment

1. Prove the following double angle identities using tan addition identity for cosine.
   a) \(\cos 2\theta = \cos^2 \theta - \sin^2 \theta\).
   b) \(\cos 2\theta = 2\cos^2 \theta - 1\).
   c) \(\cos 2\theta = 1 - 2\sin^2 \theta\).
2. Prove the identity \( \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos 2\theta \).

3. Solve the following equations for \( 0 \leq x \leq 2\pi \).
   
   a) \( \cos 2x + \cos x = 0 \)  
   b) \( \cos 2x = 1 - 2 \sin x \)  
   c) \( 2 \cos^2 \frac{x}{2} - 1 = 0 \)  
   
   d) \( \sin 2x + \cos x = 0 \)  
   e) \( \cos 2x - \sin x = 0 \)
4. The diagram shows the graphs of two trig functions 
\( y = 2 \cos 2x \) and \( y = \sin x - 1 \) for \( 0 \leq x \leq 2\pi \).

a) Describe how to use the graphs to solve the equation 
\( 2 \cos 2x - \sin x + 1 = 0 \), where \( 0 \leq x \leq 2\pi \).
Mark these points with **DOTS** on the grid.

b) Find the solution to the equation \( 2 \cos 2x - \sin x + 1 = 0 \), where \( 0 \leq x \leq 2\pi \),
by algebraic means, to the nearest hundredth.

c) Describe how to use the graphs to solve the equation 
\( 2 \cos 2x (\sin x - 1) = 0 \), where \( 0 \leq x \leq 2\pi \).
Mark these points with a **SQUARE** on the grid.

d) Solve algebraically the equation \( 2 \cos 2x (\sin x - 1) = 0 \), where \( 0 \leq x \leq 2\pi \).
Give the answers as exact values.
5. Express each of the following in terms of a single trigonometric function.
   
   a) \(2 \sin \frac{1}{2}x \cos \frac{1}{2}x\)  
   b) \(\cos^2 2A - \sin^2 2A\)  
   c) \(1 - 2 \sin^2 3x\)

6. Use a double angle identity to simplify and evaluate
   
   a) \(2 \sin 15^\circ \cos 15^\circ\)  
   b) \(\cos^2 \frac{\pi}{12} - \sin^2 \frac{\pi}{12}\)  
   c) \(\sin \frac{5\pi}{12} \cos \frac{5\pi}{12}\)

Multiple Choice
7. The expression \(\frac{\cos^2 \frac{3}{2}x - \sin^2 \frac{3}{2}x}{\sin \frac{3}{2}x \cos \frac{3}{2}x}\) is equivalent to
   
   A. \(\cos \frac{3}{2}x - \sin \frac{3}{2}x\)
   B. \(\cot 3x\)
   C. \(2 \cot 3x\)
   D. \(2 \csc 3x\)

Numerical Response
8. If \(a \cos^2 \frac{\pi}{8} - a \sin^2 \frac{\pi}{8} = 4\sqrt{2}\), the value of \(a\), to the nearest tenth, is _____.
A function whose graph resembles the sine or cosine curve is called a sinusoidal function. The graph of a sinusoidal function is called a sinusoidal graph. Many periodic phenomena have sinusoidal graphs, e.g., the time of sunrise as a function of the day of the year, the height of a chair of a Ferris wheel as a function of time, the depth of the ocean due to changing tides as a function of time, etc.

In this lesson the equation of the sinusoidal function will be given. In the next lesson we will derive the equation of the sinusoidal function from given information.

Most of the equations used will be functions of time and the variable used will be $t$. The period of the graph will be in time units. Graphical methods will be used to solve problems and determining a suitable window is an essential feature of the solution.

Class Ex. #1

The minimum depth, $d$ metres, of water in a harbour, $t$ hours after midnight, can be approximated by the function $d(t) = 12 + 5 \cos 0.5t$, where $0 \leq t \leq 24$.

a) Determine the maximum and minimum depths of water in the harbour.

b) Determine the period of the function.

c) Write a suitable window which can be used to display the graph of the function.

d) What is the depth of water, to the nearest tenth of a metre at 2:00 a.m.?

e) A ship which requires a minimum of 8.5 metres of water is in harbour at midnight. By what time, to the nearest minute, must it leave to prevent grounding?

f) What is the next time, to the nearest necessary minute, that the ship can return to the harbour?
In a certain town in British Columbia, the time of sunrise for any day can be found using the formula

\[ t = -1.79 \sin \left( \frac{2\pi (d - 78)}{365} \right) + 6.3 \]

where \( t \) is the time in hours after midnight and \( d \) is the number of the day in the year.

a) Write a suitable window which can be used to display the graph of the function.

b) Use the formula to determine, to the nearest minute, when the sun rose on May 7, the 127th day of the year.

c) Determine on which days of the year the sun rose at 7 a.m.

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**Assignment**

1. The alarm in a noisy factory is a siren whose volume, \( V \) decibels fluctuates so that \( t \) seconds after starting, the volume is given by the function \( V(t) = 18 \sin \frac{\pi}{15} t + 60 \).

   a) What are the maximum and minimum volumes of the siren?

   b) Determine the period of the function.

   c) Write a suitable window which can be used to display the graph of the function.

   d) After how many seconds, to the nearest tenth, does the volume first reach 70 decibels?

   e) The background noise level in the factory is 45 decibels. Between which times, to the nearest tenth of a second, in the first cycle is the alarm siren at a lower level than the background noise?

   f) For what percentage, to the nearest per cent, of each cycle is the alarm siren audible over the background factory noise?
2. A top secret satellite is launched into orbit from a remote island not on the equator. When the satellite reaches orbit, it follows a sinusoidal pattern that takes it north and south of the equator, (i.e. the equator is used as the horizontal axis). Twelve minutes after it is launched it reaches the farthest point north of the equator. The distance north or south of the equator can be represented by the function

\[ d(t) = 5000 \cos \left( \frac{\pi}{35} (t - 12) \right) \]

where \( d(t) \) is the distance of the satellite north of the equator \( t \) minutes after being launched.

a) How far north or south of the equator is the launch site? Answer to the nearest km.

b) Is the satellite north or south of the equator after 20 minutes? What is this distance to the nearest kilometre?

c) When, to the nearest tenth of a minute, will the satellite first be 2500 km south of the equator?

3. The height of a tidal wave approaching the face of the cliff on an island is represented by the equation

\[ h(t) = 7.5 \cos \left( \frac{2\pi}{9.5} t \right) \]

where \( h(t) \) is the height, in metres, of the wave above normal sea level \( t \) minutes after the wave strikes the cliff.

a) What are the maximum and minimum heights of the wave relative to normal sea level?

b) What is the period of the function?

c) How high, to the nearest tenth of a metre, will the wave be, relative to normal sea level, one minute after striking the cliff?

d) Normal sea level is 6 metres at the base of the cliff.

i) For what values of \( h \) would the sea bed be exposed?

ii) How long, to the nearest tenth of a minute, after the wave strikes the cliff does it take for the sea bed to be exposed?

iii) For how long, to the nearest tenth of a minute, is the sea bed exposed?
4. The depth of water in a harbour can be represented by the function

\[ d(t) = -5 \cos \left( \frac{\pi}{6} t \right) + 16.4 \]

where \( d(t) \) is the depth in metres and \( t \) is the time in hours after low tide.

a) What is the period of the tide?

b) A large cruise ship needs at least 14 metres of water to dock safely. For how many hours per cycle, to the nearest tenth of an hour, can a cruise ship dock safely?

5. A city water authority determined that, under normal conditions, the approximate amount of water, \( W(t) \), in millions of litres, stored in a reservoir \( t \) months after May 1, 2003, is given by the formula \( W(t) = 1.25 - \sin \left( \frac{\pi}{6} t \right) \).

a) Sketch the graph of this function over the next three years.

b) The authority decided to carry out the following simulation to determine if they had enough water to cope with a serious fire.

“If, on November 1, 2004, there is a serious fire which uses 300 000 litres of water to bring under control, will the reservoir run dry if water rationing is not imposed?”

i) Explain how to use the graph in a) to solve the problem.

ii) Will the reservoir run dry if water rationing is not imposed? If so, in what month will this occur?

**Answer Key**

1. a) \[ \text{max} = 78 \text{ dB}, \text{min} = 42 \text{ dB}, \]  
   b) 30 s  
   c) \( x: [0, 40, 5] \)  
   d) 2.8 s  
   e) 19.7 s - 25.3 s  
   f) 81%  

2. a) 2369 km north  
   b) 3765 km north  
   c) 35.3 minutes  

3. a) \[ \text{max} = 7.5 \text{ m}, \text{min} = -7.5 \text{ m} \]  
   b) 9.5 min  
   c) 5.9 m  
   d) i) \( h \leq -6 \)  
   ii) 3.8 min  
   iii) 1.9 min  

4. a) 12 h  
   b) 7.9 h  

5. b) i) After \( t = 18 \) move the graph down 0.3 units. If the graph falls below the \( t \) axis, the reservoir will run dry.  
   or  
   Draw the line \( y = 0.3 \). If the line intersects the graph, the reservoir will run dry.  
   ii) In month 26, i.e. July, 2005
**Trigonometry - Equations, Identities, and Modelling Lesson #10: Modelling**

**Warm-Up #1**  
**Introduction**

If the previous lesson we were asked to solve problems when we were given the equation of a sinusoidal function.

In this lesson we will derive the equation of the sinusoidal function from a graph.

**Warm-Up #2**  
**Review**

The sinusoidal wave shown has a maximum value of 50 and a minimum value of 10. Write the equation of the sinusoidal wave in the form \( h(t) = a \sin (b(t - c)) + d \), where \( a > 0 \).

---

**Class Ex. #1**

A nail is caught in the tread of a rotating tire at point \( N \) in the following sketch.

The tire has a diameter of 50 cm and rotates at 10 revolutions per minute. After 4.5 seconds the nail touches the ground.

**a)** Use the information given to write a scale for each axis.

**b)** Determine the equation for the height of the nail as a function of time in the form \( h(t) = a \sin bt + d \), where \( a > 0 \).

**c)** How far, to the nearest tenth of a centimetre, is the nail above the ground after 6.5 seconds?
The first Ferris wheel ever built was created by a bridge builder by the name of George W. Ferris in 1893. The diameter of the wheel was approximately 76 metres and the maximum height of the Ferris Wheel was approximately 80 metres. It had 36 carts on the wheel, with each cart able to hold approximately 60 people.

a) If the wheel rotates every three minutes, draw a graph which represents the height of a cart, in metres, as a function of time in minutes. Assume that the cart is at its highest position at $t = 0$. Show three complete cycles.

b) Determine the equation of the graph in the form $h(t) = a \cos bt + d$.

c) How high is the cart 5 minutes after the wheel starts rotating? Answer to the nearest metre.

d) How many seconds after the wheel starts rotating does the cart first reach 10 metres from the ground? Answer to the nearest second.
Assignment

1. The graph shows the height, \( h \) metres, above the ground over time, \( t \), in seconds that it takes a person in a chair on a ferris wheel to complete two revolutions. The minimum height of the ferris wheel is 2 metres and the maximum height is 20 metres.

\[
\begin{align*}
\text{h} & \quad \text{t} \\
20 & \quad 10 \\
15 & \quad 20 \\
10 & \quad 30 \\
5 & \quad 40 \\
& \quad 50 \\
& \quad 60
\end{align*}
\]

a) How far above the ground is the person as the wheel starts rotating?

b) If it takes 16 seconds for the person to return to the same height, determine the equation of the graph in the form \( h(t) = a \sin bt + d \).

c) Find the distance the person is from the ground, to the nearest tenth of a metre, after 30 seconds?

d) How long from the start of the ride does it take for the person to be at a height of 5 metres? Answer to the nearest tenth of a second.
2. A Ferris wheel ride can be represented by a sinusoidal function. A Ferris wheel at Westworld Theme Park has a radius of 15 m and travels at a rate of six revolutions per minute in a clockwise rotation. Ling and Lucy board the ride at the bottom chair from a platform one metre above the ground.

a) Sketch three cycles of a sinusoidal graph to represent the height Ling and Lucy are above the ground, in metres, as a function of time, in seconds.

b) Determine the equation of the graph in the form \( h(t) = a \cos [b(t - c)] + d \).

c) If the Ferris wheel does not stop, determine the height Ling and Lucy are above the ground after 28 seconds. Give answer to the nearest tenth of metre.

d) How long after the wheel starts rotating do Ling and Lucy first reach 12 metres from the ground? Give answer to the nearest tenth of a second.

e) How long does it take from the first time Ling and Lucy reach 12 metres until they next reach 12 metres from the ground? Give answer to the nearest second.
3. Andrea, a local gymnast, is doing timed bounces on a trampoline. The trampoline mat is 1 metre above ground level. When she bounces up, her feet reach a height of 3 metres above the mat, and when she bounces down her feet depress the mat by 0.5 metres. Once Andrea is in a rhythm, her coach uses a stopwatch to make the following readings:

- At the highest point the reading is 0.5 seconds.
- At the lowest point the reading is 1.5 seconds.

a) Sketch two periods of the graph of the sinusoidal function which represents Andrea’s height above the ground, in metres, as a function of time, in seconds.

b) How high was Andrea above the mat when the coach started timing?

c) Determine the equation of the graph in the form \( h(t) = a \sin bt + d \).

d) How high, to the nearest tenth of a metre, was Andrea above the ground after 2.7 seconds?

e) How high, to the nearest tenth of a metre, was Andrea above the mat after 17 seconds?

f) How long after the timing started did Andrea first touch the mat?  
   Answer to the nearest tenth of a second.
4. Consider the following information for a town in Saskatchewan for a leap year of 366 days:
   
   - The latest sunrise time is at 09:00 on December 21 (day 356)
   - The earliest sunrise time is at 03:30 on June 21 (day 173)
   - There is NO daylight saving time in Saskatchewan
   - The sunrise times vary sinusoidally with the day of the year.

   a) Write a sinusoidal equation which relates the time of sunrise, \( t \), to the day of the year, \( d \).

   b) Use the equation to determine what time, to the nearest minute, the sun rises on March 11.

   c) Determine the average time the sun rises throughout the year.

   d) How many days of the year does the sun rise before 6 a.m.?
Use the following information to answer the next two questions

The graph below shows how the number of hours \( (h) \) of daylight in a European city changes during the year.

\[
h = \frac{1}{3} \left[ 35 + 7 \sin \left( \frac{360x}{365} \right) \right]
\]

5. Mid-winter is the day with the least hours of daylight. The number of hours of daylight, to the nearest tenth of an hour, that there will be on mid-winter’s day is _____.

6. The number of days after April 21 that mid-winter occurs is _____.
Answer Key

1. a) 11 metres  b) \( h(t) = 9 \sin \left( \frac{\pi}{16} t \right) + 11 \)  c) 7.6 metres  d) 19.7 seconds

2. b) \( h(t) = 15 \cos \frac{\pi}{5} (t - 5) + 16 \)  c) 11.4 metres  d) 2.1 seconds  e) 6 seconds

3. b) 1.25 metres  c) \( h(t) = 1.75 \sin \pi t + 2.25 \)  d) 3.7 metres  e) 1.3 metres  f) 1.3 seconds

4. a) \( t = 2.75 \cos \left( \frac{\pi}{183} (d + 10) \right) + 6.25 \)  b) 06:45  c) 06:15  d) 173 5. 9.3 6. 274
Blanche is shopping for a car. The dealer tells her that he has 3 different models and that each model is available in 5 different colours.

a) Use the tree diagram to determine how many different choices Blanche has.

b) Look at the number of choices there are for the model and the number of choices there are for the colour. How do you use these numbers to arrive at the answer in a)?

The Fundamental Counting Principle

Consider a task made up of several stages. If the number of choices for the first stage is \( a \), the number of choices for the second stage is \( b \), the number of choices for the third stage is \( c \), etc., then the number of ways in which the task can be completed is \( a \times b \times c \times \ldots \).

This is called the fundamental counting principle.

A toy manufacturer makes a wooden toy in three parts:

Part 1: the top part may be coloured red, white or blue,
Part 2: the middle part may be orange or black,
Part 3: the bottom part may be yellow, green, pink or purple.

Determine how many different coloured toys can be produced using:

a) a tree diagram 

b) the fundamental counting principle
Determine the number of distinguishable four letter arrangements that can be formed from the word **ENGLISH** if:

a) letters can be repeated?

b) no letters are repeated and:

i) there are no further restrictions?  
ii) the first letter must be E?

iii) the “word” must contain G?  
iv) the first and last letters must be vowels?

A Math 30 quiz consists of eight multiple choice questions. Each question has four choices A, B, C or D. How many different sets of answers are possible?

The telephone numbers allocated to subscribers in a rural area consist of one of the following:

- the digits 345 followed by any three further digits, or,
- the digit 2 followed by one of the digits 1 to 5 followed by any three further digits.

How many different telephone numbers are possible?

Car number plates in an African country consist of a letter other than I or O followed by three digits, the first of which cannot be zero, followed by any two letters which are not repeated. How many different car number plates can be produced?
Class Ex. #6

Consider the digits 2, 3, 5, 6, 7 and 9.

a) If repetitions are not permitted, how many 3-digit numbers can be formed?

b) How many of these are

i) less than 400? 

ii) even?

iii) odd?

iv) multiples of 5?

Class Ex. #7

Consider a six-digit numeral. (Note: 022713 is classified as the 5-digit numeral 22713).

a) How many odd six-digit numerals have no repeating digits?

b) How many even six-digit numerals have no repeating digits?

Assignment

1. A football team has the following kit: jersey: red or black, pants: white, red or black, socks: red or white

If the team plays in a different uniform each week, for how many weeks can it play before it has to repeat a previous uniform?

2. The score at the end of the second period of a hockey game is: Canucks 6 Oilers 3.

How many different possibilities are there for the score at the end of the first period?

3. With the new renovations completed at Prestwick High School, there will be seven entrances. In how many different ways can a student coming for Math tutorials;

a) enter the school and exit through a different entrance?

b) enter and exit through any entrance?

c) enter and exit through the same entrance?
4. How many ways are there of getting from A to C in each diagram, passing through each point at most once?

Answer to Diagram 1

Answer to Diagram 2

Answer to Diagram 3

5. Find the number of four letter “words” that can be formed from the letters of the word PRODUCE if;

a) each letter can only be used once:

b) each letter can only be used once and the “word” must:
   i) contain only consonants?
   ii) begin and end with a consonant?

   iii) begin with a vowel?
   iv) contain the letter P?

   v) begin with D and end with a vowel?

6. In a class of 30 students, how many ways are there of awarding;

a) a first prize, a second prize and a third prize in Mathematics?

b) a Mathematics prize, a Chemistry prize and a Physics prize?
   (Assume that each of the students is capable of winning any of the prizes.)

7. How many even four digit numerals have no repeated digits?
8. A vehicle license plate consists of 3 letters followed by 3 digits. How many different license plates are possible if:
   a) there are no restrictions on the letters or digits used?
   b) no letters may be repeated?
   c) the first digit cannot be zero and no digits can be repeated?

9. a) How many different three-digit numerals can be formed from the digits 1, 5 and 8 if the digits cannot be repeated?
   b) How many different three-digit numerals can be formed using the digits 1, 3, 5, 7 and 9 if the digits may be repeated?
   c) How many four-digit numerals can be formed from the digits 0, 2 and 3 if the digits may be repeated? (Note: 0223 is classified as the 3-digit numeral 223).
   d) How many different non-zero numerals are possible using some or all of the numerals 0, 1, 2, and 3 if the digits cannot be repeated?

10. How many different sums of money can be made from two pennies, four nickels, two quarters, and five dollar coins?

11. Mr. and Mrs. McDonald want a family picture taken with their children, Hamish, Flora and James. In how many different ways can all five line up in a straight line for the picture if;
   a) there are no restrictions?
   b) the parents must be at either end of the line?
   c) baby James must be in the middle?
   d) the children alternate with the adults?
12. Determine the number of distinguishable four letter arrangements that can be formed from the word **PRODUCT** given the following restrictions:

   a) no restrictions
   b) the "word" begins with **PR**?
   c) the "word" has two vowels in the middle?
   d) the "word" has two consonants in the middle?

13. Ocean going ships have use coloured flags hung vertically for signalling. By changing the order of the coloured flags, the ships can send out different signals. If ships carry six different coloured flags, one flag of each colour, how many different signals are possible if;

   a) all six flags are used?
   b) four flags are used?
   c) at least two flags are used?

14. Sandra is taking an examination which consists of two parts, A and B, with the following instructions.

   - Part A consists of three questions and the student must do two.
   - Part B consists of four questions and the student must do two.
   - Part A must be completed before starting Part B.
   - At the end of the exam the student has to list the order in which she attempted the questions.

   The number of different possible orders is ______.

**Answer Key**

1. 12
2. 28
3. a) 42  b) 49  c) 7

4. Diagram 1 → 15  Diagram 2 → 15  Diagram 3 → 11

5. a) 840  b) 1  i) 24  ii) 240  iii) 360  iv) 480  v) 60

6. a) 24360  b) 27000

7. 2296

8. a) 17,576,000  b) 15,600,000  c) 11,389,248

9. a) 6  b) 125  c) 54  d) 48

10. 269

11. a) 120  b) 12  c) 24  d) 12

12. a) 840  b) 20  c) 40  d) 400

13. a) 720  b) 360  c) 1950

14. 72
Permutations and Combinations Lesson #2: Factorial Notation and Permutations

Factorial Notation

Consider how many ways there are of arranging 6 different books side by side on a shelf.

In this example we have to calculate the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$

In mathematics this product is denoted by $6!$ ("6 factorial" or "factorial 6")

In general $n! = n(n-1)(n-2)(n-3) \ldots (3)(2)(1)$, where $n \in \mathbb{W}$.

Warm-Up #1

a) Use the factorial key on a calculator to evaluate the following.

$6! = \underline{\hspace{2cm}}$  $9! = \underline{\hspace{2cm}}$

b) To simplify a quantity like $\frac{10!}{7!}$ there are a variety of approaches:

i) By using the factorial key on your calculator

$\frac{10!}{7!} = \frac{3,628,800}{5,040} = 720$ or $\frac{10!}{7!} = 720$

ii) By Cancellation

$\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 720$

Class Ex. #1

Find the value of:

a) $\frac{43!}{40!}$

b) $\frac{37!}{33! \cdot 4!}$

Class Ex. #2

Simplify the following expressions

a) $\frac{n!}{(n-2)!}$

b) $\frac{(n+3)!}{n!}$

c) $\frac{n!}{n(n-1)}$

Complete Assignment Questions #1 - #5
An internet site’s access code consists of three digits. Knowing the three digits is not enough to access the site. The digits have to be entered in the correct order. The order of the arrangement of the digits is important.

Joe cannot remember the access code, except that it contains the digits 3, 5 and 7.

- List all the arrangements of these three digits that Joe could use to determine the access code.

There are six possible arrangements to consider and only one of them will access the site. This type of arrangement, where the order is important, is called a permutation.

**Permutations**

An arrangement of a set of objects in which the order of the objects is important is called a permutation.

**Permutations of “n” different objects taken all at a time**

- List all the arrangements of the letters of the word CAT.
- Write the number of permutations in factorial notation.

This is an example of the following general rule:

The number of permutations of “n” different objects taken all at a time is \( n! \)

How many permutations are there of the letters of the word:

a) REGINA  
b) KELOWNA
Permutations of “n” different objects taken “r” at a time \((r \leq n)\)

- Use the fundamental counting principle to determine how many three letter arrangements can be made from the letters of the word GRAPHITE.

In the example above we have found the number of permutations of 8 \((n)\) objects taken 3 \((r)\) at a time. This is denoted by \(8P_3\).

\[
8P_3 = 8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5!}{5!} = \frac{8!}{(8 - 3)!}
\]

This is an example of the following general rule:

The number of permutations of “n” different objects taken “r” at a time is

\[
\text{nPr} = \frac{n!}{(n - r)!}
\]

This formula is on the formula sheet

Use the \(nPr\) key on a calculator to evaluate \(8P_3\). Verify using factorials.

**Defining 0!**

If we replace \(r\) by \(n\) in the above formula we get the number of permutations of \(n\) objects taken \(n\) at a time. This we know is \(n!\)

\[
nP_n = n! = \frac{n!}{(n - n)!} = \frac{n!}{0!}
\]

For this to be equal to \(n!\) the value of 0! must be 1.

0! is defined to have a value of 1.

In a South American country, vehicle license plates consist of any 2 different letters followed by 4 different digits. Find how many different license plates are possible by:

a) the fundamental counting principle  
b) permutations
Solve for $n$ in the equation $nP_4 = 28 \times n \cdot P_2$

• In many cases involving simple permutations, the fundamental counting principle can be used in place of the permutation formulas.

Complete Assignment Questions #6 - #13

Assignment

1. Express as single factorials. (No work required.)
   
   a) $6 \times 5 \times 4 \times 3 \times 2 \times 1$
   
   b) $9 \times 8 \times 7 \times 6!$
   
   c) $(n + 2)(n + 1)n(n - 1) \ldots \times 3 \times 2 \times 1$

2. Find the value of the following:
   
   a) $10!$
   
   b) $\frac{8!}{4!}$
   
   c) $\frac{15!}{10! \cdot 5!}$
   
   d) $\frac{25!}{21! \left( \frac{7!}{11!} \right)}$

3. Simplify each expression.
   
   a) $\frac{n!}{n}$
   
   b) $\frac{(n - 3)!}{(n - 2)!}$
   
   c) $\frac{(n + 1)!}{(n - 1)!}$
   
   d) $\frac{(3n)!}{(3n - 2)!}$
4. Express as a quotient of factorials.
   a) $9 \times 8 \times 7 \times 6$  
   b) $20 \times 19 \times 18$  
   c) $(n + 2)(n + 1)n$

5. Solve the equation.
   a) $\frac{(n + 1)!}{n!} = 6$  
   b) $(n + 1)! = 6(n - 1)!$  
   c) $\frac{(n + 2)!}{n!} = 12$  
   d) $\frac{(n + 1)!}{(n - 2)!} = 20(n - 1)$

6. How many arrangements are there of the letters:
   a) DOG  
   b) DUCK  
   c) SANDWICH  
   d) CANMORE ?

7. How many five-digit numbers can be made from the digits 2, 3, 4, 7 and 9 if no digit can be repeated?

8. If $_nP_r$ is the number of ways that $n$ objects can be arranged $r$ at a time, explain why $_nP_0 = 1$.

9. Use a permutation formula to determine how many arrangements there are of
   a) two letters from the word GOLDEN  
   b) three letters from the word CHAPTERS  
   c) four letters from the word WEALTH
10. How many numbers can be made from the digits 2, 3, 4, and 5 if no digit can be repeated? (Hint: consider 4 cases - four-digit numbers, three-digit numbers, two-digit numbers, one-digit numbers.)

11. Solve each equation, where $n$ is an integer.
   a) $\frac{n!}{84} = n - 2P_{n-4}$  
   b) $nP_4 = 8(n - 1P_3)$

12. In a ten-team basketball league, each team plays every other team twice, once at home and once away. The number of games that are scheduled is
   
   A. 45  
   B. 90  
   C. 100  
   D. 180

13. In a competition on the back of a cereal packet, seven desirable qualities for a kitchen (eg. spaciousness, versatility etc.) must be put in order of importance. The number of different entries that must be completed in order to ensure a winning order is _____.

**Answer Key**

1. a) 6!  
   b) 9!  
   c) $(n + 2)!$  
   2. a) 3 628 800  
   b) 1680  
   c) 3003  
   d) $\frac{115}{3}$

3. a) $(n - 1)!$  
   b) $\frac{1}{n - 2}$  
   c) $n(n + 1)$  
   d) $3n(3n - 1)$  
   4. a) $\frac{9!}{5!}$  
   b) $\frac{20!}{17!}$  
   c) $\frac{(n + 2)!}{(n - 1)!}$

5. a) $n = 5$  
   b) $n = 2$  
   c) $n = 2$  
   d) $n = 4$  

6. a) 6  
   b) 24  
   c) 40 320  
   d) 5040

7. 120  
8. $\gamma_7 = \frac{7!}{(7 - 0)!} = \frac{7!}{7!} = 1$  
9. a) 30  
   b) 336  
   c) 360

10. 64  
11. a) $n = 7$  
    b) $n = 8$  
12. B  
13. 5040
Permutations and Combinations Lesson #3: Permutations with Restrictions; Permutations with Repetitions

Permutations with Restrictions

In many problems restrictions are placed on the order in which objects are arranged. In this type of situation deal with the restrictions first.

Class Ex. #1
In how many ways can all of the letters of the word ORANGES be arranged if:

a) there are no further restrictions?  
b) the first letter must be an N?

c) the vowels must be together in the order O, A, and E?

Class Ex. #2
In how many of the arrangements of the letters of the word BRAINS are the vowels together?

Class Ex. #3
Find the number of permutations of the letters in the word KITCHEN if:

a) the letters K, C, and N must be together but not necessarily in that order  
b) the vowels must not be together

Class Ex. #4
In how many ways can 3 girls and 4 boys be arranged in a row if no two people of the same gender can sit together?

Class Ex. #5
Six actors and eight actresses are available for a play with four male roles and three female roles. How many different cast lists are possible?
How many odd five-digit numbers can be made with the following digits if no digits can be repeated?

a) 1, 2, 3, 4, 5, 6 and 7
b) 0, 2, 3, 4, 5, 6 and 7

### Complete Assignment Questions #1 - #7

#### Permutations With Repetitions

In the previous section, the objects in each set were all different. But what happens when there are letters which repeat within the same word? To examine this scenario, consider the following four letter permutations of a word without repetitive letters, ROSE. Notice there are \(4!\), or 24 different arrangements.

ROSE
REOS
OSRE
SROE
SERO
EORS

ROES
RESO
OSER
SREO
SEOR
EOSR

RSEO
ORSE
OERS
SORE
EROS
ESRO

RSEO
ORES
OESR
SOER
EROS
ESOR

Now, if we change the E in ROSE to a S, we get ROSS, a word with two letters which are repeating. If we change all the E’s in the above list to S’s, we will get all the arrangements for ROSS as shown in the list below.

ROSS
RSOS
OSRS
SROS
SSRO
SOR

ROSS
RSSO
OSSR
SSRS
SSOR
SOSR

RSOS
ORSS
OSRS
SORS
SROS
SSRO

RSSO
ORSS
OSSR
SORS
SRSO
SSOR

Notice that the 24 original arrangements now become 12 different arrangements - 12 matching pairs of 2 four-letter arrangements. Arrangements like ROSE and ROES from the first list both become ROSS in the second list and count as only one arrangement.

There are \(\frac{1}{2}\) or \(\frac{1}{2!}\) as many permutations of ROSS as there are of ROSE.

Hence, the number of permutations of ROSS is \(\frac{4!}{2!}\), or 12.

If we change the O and E in ROSE to S, we get RSSS, a “word” with three repeating letters, with the arrangements shown below.

RSSS
RSSS
SSRS
SRSS
SSRS
SSRS

RSSS
RSSS
SSSR
SRSS
SSSR
SSSR

RSSS
SRSS
SSSR
SSRS
SRSS
SSSR

RSSS
SRSS
SSSR
SSRS
SRSS
SSSR

Notice the 24 original arrangements now become 4 different arrangements - 4 matching sets of 6 four-letter arrangements. Arrangements like ROSE, ROES, RSOE, RSEO, REOS, and RESO from the first list all become ROSS in the last list and count as only one arrangement of RSSS.

There are \(\frac{1}{6}\) or \(\frac{1}{3!}\) as many permutations of RSSS as there are of ROSE.

Hence, the number of permutations of RSSS is \(\frac{4!}{3!}\), or 4.
There is a pattern in the above lists. If a letter appears twice in a word, we divide the total number of arrangements by 2!. If a letter appears three times in a word we divide the total number of arrangements by 3!.

The following formula gives the number of permutations when there are repetitions.

The number of permutations of \( n \) objects, where \( a \) are the same of one type, \( b \) are the same of another type, and \( c \) are the same of yet another type, can be represented by the expression below:

\[
\frac{n!}{a!b!c!}
\]

---

**Class Ex. #7**

Find the number of permutations of the letters of the word:

a) VANCOUVER  

b) MATHEMATICAL

---

**Class Ex. #8**

How many arrangements of the word POPPIES can be made under each of the following conditions?

a) without restrictions  
b) if each arrangement begins with a P

c) if the first two letters are P  
d) if all the P’s are to be together

e) if the first letter is P and the next one is not P

---

**Class Ex. #9**

Brett bought a carton containing 10 mini boxes of cereal. There are 3 boxes of Corn Flakes, 2 boxes of Rice Krispies, 1 box of Coco Pops, 1 box of Shreddies, and the remainder are Raisin Bran. Over a ten day period Brett plans to eat the contents of one box of cereal each morning.

How many different orders are possible if on the first morning he has Raisin Bran?

---

**Complete Assignment Questions #8 - #15**
Assignment

1. How many arrangements could be made of the word:
   a) **FATHER** if F is first?  
   b) **UNCLE** if C is first and L is last?
   c) **DAUGHTER** if UG is last?  
   d) **MOTHER** if the vowels are first and last?

2. How many arrangements of the following words can be made if all the vowels must be kept together?
   a) **FATHER**  
   b) **DAUGHTER**  
   c) **UNCLE**  
   d) **EQUATION**

3. Find the number of different arrangements of the letters in the word **ANSWER** under each condition:
   a) without restrictions  
   b) that begin with an S  
   c) that begin with a vowel and end with a consonant  
   d) that have the three letters A, N, and S adjacent and in the order ANS  
   e) that have the three letters A, N, and S adjacent but not necessarily in that order
4. How many even four-digit numbers can be made from the digits 0, 2, 3, 4, 5, or 7 if no digit can be repeated?

5. Ann, Brian, Colin, Diane and Eric go to watch a movie and sit in 5 adjacent seats. In how many ways can this be done under each condition?
   a) without restrictions? 
   b) if Brian sits next to Diane?
   c) if Ann refuses to sit next to Eric?

6. In how many ways can four adults and five children be arranged in a single line under each condition?
   a) without restrictions 
   b) if children and adults are alternated
   c) if the adults are all together and the children are all together 
   d) if the adults are all together

7. Fifteen rugby players line up for a team picture, with seven players in the front row and eight players in the back row. How many different arrangements are possible under each conditions? Express the answer in factorial notation.
   a) without restrictions 
   b) if the captain is in the middle of the front row

8. How many different arrangements can be made using all of the letters of each word?
   a) VICTORIA 
   b) ABBOTSFORD 
   c) LILLOOET 
   d) OSOYOOS
9. Find the number of arrangements of the letters of the word \textbf{TATTOO} under each condition:
   \(\text{a) begins with a T} \quad \text{b) begins with two T’s} \quad \text{c) begins with three T’s} \)
   \(\text{d) begins with one T and the next letter is not a T} \quad \text{e) begins with exactly two T’s} \)

10. A town has 10 streets running from north to south and 8 avenues running from west to east. A man wishes to drive from the extreme south-west intersection to the extreme north-east intersection, moving either north or east along one of the streets or avenues. Find the number of routes he can take. (This question will be solved in a different manner in a lesson 7 of this unit).

11. How many permutations are there of the letters of the word \textbf{MONOTONOUS} under each condition?
   \(\text{a) without restrictions} \quad \text{b) if each arrangement begins with a T} \)
   \(\text{c) if each arrangement begins with an O} \quad \text{d) if the four O’s are to be together} \)

12. A race at the Olympics has 8 runners. In how many orders can their countries finish if there are:
   \(\text{a) 2 Canadian, 1 Russian, 1 German, 1 South African and 3 American athletes.} \)
   \(\text{b) 1 Canadian, 2 British, 2 Ethiopian, 1 Algerian and 2 Kenyan runners} \)
13. Naval signals are made by arranging coloured flags in a vertical line and the flags are then read from top to bottom. How many signals using six flags can be made if you have:

a) 3 red, 1 green and 2 blue flags?  

b) 2 red, 2 green and 2 blue flags?

c) unlimited supplies of red, green and blue flags?

14. The number of different arrangements can be made using all the letters of the word SASKATOON is

A. 720  
B. 45 360  
C. 362 880  
D. 725 760

15. The number of different ways that seven basketball players can be seated on a bench so that two specified players are always sitting side by side is _____.
Permutations and Combinations Lesson #4: Combinations Part 1

Warm-Up #1

As part of the grade 12 English course, students are required to read the following three books in a three month period:


a) Due to previous late returns Steve is only allowed to sign out one English book from the school library per month. List all the different orders in which Steve could sign out the books.

b) Tariq is allowed to sign out all three books at the same time. How many different ways can he sign out all three books at the same time?

Part a) is an example of a permutation where the order is important. Part b) is an example of a combination where the order is NOT important.

Combinations

A selection of a set of objects in which the order of the selection is NOT important is called a combination.

Warm-Up #2

Suppose that the students in Warm-Up #1 were required to read only two of the books.

a) Complete the table to show the number of ways in which Steve and Tariq could do this:

<table>
<thead>
<tr>
<th>Steve (Permutations)</th>
<th>Tariq (Combinations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Grapes of Wrath, The Wars</td>
<td>The Grapes of Wrath, The Wars</td>
</tr>
<tr>
<td>The Grapes of Wrath, The Bean Trees</td>
<td></td>
</tr>
<tr>
<td>The Wars, The Grapes of Wrath</td>
<td></td>
</tr>
</tbody>
</table>

b) Complete the following statement:

• The number of combinations is equal to the number of permutations divided by ____ or ____ factorial.
Warm-Up #3

Five students, Al, Byron, Colin, Dave and Eric take part in a cross country race to represent their school.

a) Suppose the winner of the race wins $50, the runner-up wins $25 and third place is $10. The table below shows all possible ways in which the three prizes could be awarded to the five participants in the race.


<table>
<thead>
<tr>
<th>ABC</th>
<th>ABD</th>
<th>ABE</th>
<th>ACD</th>
<th>ACE</th>
<th>ADE</th>
<th>BCD</th>
<th>BCE</th>
<th>BDE</th>
<th>CDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACB</td>
<td>ADB</td>
<td>AEB</td>
<td>ADC</td>
<td>AEC</td>
<td>AED</td>
<td>BDC</td>
<td>BEC</td>
<td>BDE</td>
<td>CED</td>
</tr>
<tr>
<td>BAC</td>
<td>BAD</td>
<td>BAE</td>
<td>CAD</td>
<td>CAE</td>
<td>DAE</td>
<td>CBD</td>
<td>CBE</td>
<td>DBE</td>
<td>DCE</td>
</tr>
<tr>
<td>BCA</td>
<td>BDA</td>
<td>BEA</td>
<td>CDA</td>
<td>CEA</td>
<td>DEA</td>
<td>CDB</td>
<td>CEB</td>
<td>DEB</td>
<td>DEC</td>
</tr>
<tr>
<td>CAB</td>
<td>DAB</td>
<td>EAB</td>
<td>DAC</td>
<td>EAC</td>
<td>EAD</td>
<td>DBC</td>
<td>EBC</td>
<td>EBD</td>
<td>ECD</td>
</tr>
<tr>
<td>CBA</td>
<td>DBA</td>
<td>EBA</td>
<td>DCA</td>
<td>ECA</td>
<td>EDA</td>
<td>DCB</td>
<td>ECB</td>
<td>EDB</td>
<td>EDC</td>
</tr>
</tbody>
</table>

• Is this an example of permutations or combinations?
• How many ways are there to award the three prizes?

b) For participating in the cross-country race, the school has been awarded three places at a running clinic. The school coach decides to select the 3 lucky students from the ones who took part in the cross country race.

• Use the table from a) (which has been duplicated below) to circle the different ways the three students can be chosen.

<table>
<thead>
<tr>
<th>ABC</th>
<th>ABD</th>
<th>ABE</th>
<th>ACD</th>
<th>ACE</th>
<th>ADE</th>
<th>BCD</th>
<th>BCE</th>
<th>BDE</th>
<th>CDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACB</td>
<td>ADB</td>
<td>AEB</td>
<td>ADC</td>
<td>AEC</td>
<td>AED</td>
<td>BDC</td>
<td>BEC</td>
<td>BDE</td>
<td>CED</td>
</tr>
<tr>
<td>BAC</td>
<td>BAD</td>
<td>BAE</td>
<td>CAD</td>
<td>CAE</td>
<td>DAE</td>
<td>CBD</td>
<td>CBE</td>
<td>DBE</td>
<td>DCE</td>
</tr>
<tr>
<td>BCA</td>
<td>BDA</td>
<td>BEA</td>
<td>CDA</td>
<td>CEA</td>
<td>DEA</td>
<td>CDB</td>
<td>CEB</td>
<td>DEB</td>
<td>DEC</td>
</tr>
<tr>
<td>CAB</td>
<td>DAB</td>
<td>EAB</td>
<td>DAC</td>
<td>EAC</td>
<td>EAD</td>
<td>DBC</td>
<td>EBC</td>
<td>EBD</td>
<td>ECD</td>
</tr>
<tr>
<td>CBA</td>
<td>DBA</td>
<td>EBA</td>
<td>DCA</td>
<td>ECA</td>
<td>EDA</td>
<td>DCB</td>
<td>ECB</td>
<td>EDB</td>
<td>EDC</td>
</tr>
</tbody>
</table>

• Is this an example of permutations or combinations?
• How many ways are there to select the three students?

c) Complete the following statement:

• The number of combinations is equal to the number of permutations divided by _____ or _____ factorial.
Combinations of “n” different objects taken “r” at a time \( (r \leq n) \)

The Warm-Ups are examples of the following general rule:

\[
\binom{n}{r} = \frac{n^P_r}{r!} = \frac{n!}{r!(n-r)!}
\]

The number of combinations of “n” items taken “r” at a time is

\[
\binom{n}{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

This formula is on the formula sheet

- The \( nC_r \) key on a calculator can be used to evaluate combinations.

Class Ex. #1

Three students from a class of ten are to be chosen to go on a school trip.

a) In how many ways can they be selected?
   Write the answer in factorial notation and evaluate.

b) Confirm the answer in a) using the \( nC_r \) key on a calculator.

Class Ex. #2

To win the LOTTO 649 a person must correctly choose six numbers from 1 to 49. Jasper, wanting to play the LOTTO 649, began to wonder how many numbers he could make up. How many choices would Jasper have to make to ensure he had the six winning numbers?

Class Ex. #3

The Athletic Council decides to form a sub-committee of seven council members to look at how funds raised should be spent on sports activities in the school. There are a total of 15 athletic council members, 9 males and 6 females. The sub-committee must consist of exactly 3 females.

a) In how many ways can the females be chosen?

b) In how many ways can the males be chosen?

c) In how many ways can the sub-committee be chosen?

d) In how many ways can the sub-committee be chosen if Bruce, the football coach must be included?
A standard deck of 52 cards has the following characteristics:
- 4 suits (spades, clubs, diamonds, and hearts).
- Each suit has 13 cards.
- Two suits are black (spades and clubs).
- Two suits are red (diamonds and hearts).
- Face cards are considered to be Jacks, Queens, and Kings.

Poker is a card game played from a deck of 52 cards.

a) How many different five card poker hands are possible?

b) In how many of the hands in a) will there be:
   i) all diamonds?  ii) 4 black cards and 1 red card?  iii) 3 kings and 2 queens?
   iv) 3 kings?  v) four aces?  vi) 5 cards of the same suit? (called a “flush”)

Complete Assignment Questions #1 - #12

Assignment

1. Pete’s Perfect Pizza Company has 9 choices of topping available.
   a) How many different 2-topping pizzas can be made?
   b) How many different 3-topping pizzas can be made?

2. A theatre company consisting of 6 players is to be chosen from 15 actors. How many selections are possible if the company must include Mrs. Jones?

3. How many different rectangles can be formed by eight horizontal lines and three vertical lines?
4. Edinburgh High School has a twelve-member student council. A four member sub-committee is to be selected to organize dances.

   a) How many different sub-committees are possible?

   b) How many four member sub-committees are possible if the council president and vice-president must be members?

5. A basketball coach has five guards and seven forwards on his basketball team.

   a) In how many different ways can he select a starting team of two guards and three forwards?

   b) How many different starting teams are there if the star player, who plays guard, must be included?

6. Twelve face cards are removed from a deck of fifty-two cards. From the face cards, three card hands are dealt. Determine the number of distinct three card hands that are possible which include:

   a) no restrictions
   
   b) 3 kings

   c) 1 Queen and 2 kings
   
   d) only 1 Jack

7. Consider a standard deck of 52 cards. Determine the number of distinct six card hands that are possible which include:

   a) no restrictions
   
   b) only clubs
   
   c) 2 clubs and 4 diamonds

   d) no sevens
   
   e) 4 tens
   
   f) only 1 Jack and 4 Queens
8. Explain the meaning of $\binom{10}{2}$. Why does $2C_{10}$ not make sense?

9. Develop a problem where $\binom{9}{4}$ would be applicable as a solution.

10. The number of ways that an executive committee consisting of prime minister, deputy prime minister, treasurer, and secretary can be chosen from 16 student council members is

   A. $4!$
   
   B. $\frac{16!}{4!}$
   
   C. $\frac{16!}{12! 4!}$
   
   D. $\frac{16!}{12!}$

11. There are three girls and six boys on the school softball team. The team consists of a pitcher, a catcher, four infielders, and three outfielders. How many ways can the nine different positions be filled if the pitcher must be a girl and the catcher must be a boy?

   A. $3C_1 \times 6C_1 \times 7!$
   
   B. $3C_1 \times 6C_1 \times 9!$
   
   C. $3C_1 \times 6C_1$
   
   D. $\frac{9!}{3! 6!}$

12. Sarah is one of a group of eight people from which a committee of four people must be formed. The number of different committees possible if Sarah must sit on the committee is ______.

Answer Key

1. a) 36   b) 84
2. 2002
3. 84
4. a) 495   b) 45
5. a) 350   b) 140
6. a) 220   b) 4   c) 24   d) 112
7. a) 20 358 520   b) 1716   c) 55 770   d) 12 271 512   e) 1128   f) 176
8. The number of ways of selecting 2 items from 10 where the order of selection is not important. You cannot select 10 items from 2.
9. Answers may vary

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Permutations and Combinations Lesson #5: Combinations Part 2

Combinations Problems with “at least”, “at most”, etc.

The Student Council decides to form a sub-committee of five council members to look at how funds raised should be spent on the students of the school. There are a total of 11 student council members, 5 males and 6 females. How many different ways can the sub-committee consist of;

a) exactly three females?  
b) at least three females?

c) at least one female?

Consider a standard deck of 52 cards. How many different five card hands can be formed containing:

a) at least 1 red card?  
b) at most 2 kings?

c) exactly two pairs?

Complete Assignment Questions #1 - #6
A class consists of 5 girls and 7 boys. A committee is to be formed consisting of 2 girls and 3 boys. In how many ways can a teacher choose the committee if:

a) there are no further restrictions?

b) Johnny, the Principal’s son, has to be on the committee?

c) the twins, Peter and Paul, cannot both be on the committee?

\[ \begin{align*}
\text{Combinations which are Equivalent} \\
\text{Class Ex. #4}
\end{align*} \]

Jane calculated \( \binom{10}{2} \) to be 45 arrangements. She then calculated \( \binom{10}{8} \) to be 45 arrangements.

a) Explain in words why \( \binom{10}{2} = \binom{10}{8} \).

b) Use factorial notation to show that \( \binom{10}{2} = \binom{10}{8} \).

c) Give another two examples of equivalent combinations.

d) Prove the identity \( nC_r = nC_{n-r} \).
Solving for “n” in Combination Problems

During a Pee Wee hockey tryout, all the players met on the ice after the last practice and shook hands with each other. How many players attended the tryouts if there were 300 handshakes in all?

Polygons and Diagonals

Warm-Up

Consider circles with five points marked on the circumference.

a) How many triangles can be formed using these five points? Write the answer in combination notation.

b) How many lines can be drawn connecting two points on the circle? Write the answer in combination notation.

c) Sketch a regular pentagon.

i) How many lines can be drawn connecting two points on the pentagon, including the sides of the pentagon?

ii) How many of these lines are diagonals of the pentagon?

iii) Express the answer to ii) in terms of combinations.

d) How many diagonals are there in a regular octagon?

e) How many diagonals are there in a regular n sided polygon?
The number of diagonals in a regular $n$-sided polygon is

$$nC_2 - n$$

This formula is **NOT** on the formula sheet

A polygon has 65 diagonals. How many sides does it have?

---

**Assignment**

1. The Athletic Council decides to form a sub-committee of 6 council members to look at a new sports program. There are a total of 15 student council members, 6 females and 9 males. How many different ways can the sub committee consist of at most one male?

2. A group of 4 journalists is to be chosen to cover a murder trial. There are 5 male and 7 female journalists available. How many possible groups can be formed:
   a) consisting of 2 men and 2 women?
   b) consisting of at least one woman?

3. Consider a standard deck of 52 cards. How many different four card hands have;
   a) at least one black card?
   b) at least 2 kings?
   c) two pairs
   d) at most 2 clubs?
4. City Council decides to form a sub-committee of five aldermen to investigate transportation concerns. There are 4 males and 7 females. How many different ways can the sub-committee be formed consisting of at least one female member?

5. A basketball squad of 11 players is to be chosen from 17 available players. In how many ways can this be done if:
   a) Colin and Darryl must be selected?  
   b) Jeff or Brent cannot both be selected?

6. An all-night showing at a movie theatre is to consist of five movies. There are fourteen different movies available, ten disaster movies and four horror movies. How many possible schedules include:
   a) at least one horror movie?  
   b) at least four disaster movies?
   c) both “Airport Disaster” and “Halloween Horror”?

7. The number of ways that a selection of 7 students can be chosen from a class of 28 is the same as the number of ways that \( n \) students can be chosen from the same class. What is the value of \( n \)?

8. How many people are there in a class in which there are 20 ways to select a committee of three people?

9. Solve for \( n \).
   a) \( nC_3 = 84 \)  
   b) \( 11C_n = 330 \) (two answers)
   c) \( nC_7 = n+1C_8 \)
10. How many diagonals are there in each?
   a) a hexagon  b) a decagon  c) an $p$-sided polygon

11. There are eight visible points on the circle below. How many triangles can be made using these eight points?

12. If 35 quadrilaterals can be placed on a circle with a series of points on it, then how many points are on the circle?

13. After everyone had shaken hands once with everyone else in a room, there was a total of 66 handshakes. How many people were in the room?
   A. 11  B. 12  C. 33  D. 67

14. A basketball team consists of some guards and six forwards. If there are 420 ways to select two guards and three forwards to the starting line-up, then the number of guards on the team is _____ .

15. Collinear points are points which share the same straight line. The number of triangles which can be formed from 10 points if no three of the points are collinear is _____ .

16. There are 170 diagonals in a polygon. The number of sides of the polygon is _____ .

**Answer Key**

<table>
<thead>
<tr>
<th></th>
<th>1. 55</th>
<th>2. a) 210</th>
<th>b) 490</th>
<th>3. a) 255 775</th>
<th>b) 6961</th>
<th>c) 2808</th>
<th>d) 258 856</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.</td>
<td>a) 462</td>
<td>5. a) 5005</td>
<td>b) 7371</td>
<td>6. a) 1750</td>
<td>b) 1092</td>
<td>c) 220</td>
<td>7. 21</td>
</tr>
<tr>
<td>9.</td>
<td>a) 9</td>
<td>b) 4 or 7</td>
<td>c) 7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. a) 9</td>
<td>b) 35</td>
<td>c) $\frac{p(p-3)}{2}$</td>
<td></td>
<td>11. 56</td>
<td>12. 7</td>
<td>13. B</td>
<td></td>
</tr>
<tr>
<td>14. 7</td>
<td>15. 120</td>
<td>16. 20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Permutations and Combinations Lesson 6: Pascal’s Triangle and the Binomial Theorem

Warm-Up #1

Study the following expansion of \((a + b)^n\), where \(0 \leq n \leq 6\).

\[
\begin{align*}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a + b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{align*}
\]

You may want to do the expansions for yourself to verify the above results.

Properties of the Expansion \((a + b)^n\)

Use the above expansions to complete the following:

1. There are_________ terms.
2. The sum of the exponents of \(a\) and \(b\) in each term is _____ .
3. The exponents of \(a\) _______________ term by term from \(n\) to 0.
4. The exponents of \(b\) _______________ term by term from 0 to \(n\).
5. The coefficients in each expansion form a ________________________ arrangement.

Warm-Up #2

The coefficients in the above expansion can be put in a triangular array known as Pascal’s Triangle (named after Blaise Pascal who developed applications of the triangle in the seventeenth century.)

a) Use Warm-Up #1 to complete the 7th row of Pascal’s Triangle.

\[
\begin{array}{cccccccc}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

b) Use the pattern in the triangle to complete the 8th and 9th row.

\[
\begin{array}{cccccccc}
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\end{array}
\]

c) Complete the following expansions:

\[
(a + b)^7 = a^7 + \\
(a + b)^8 = 
\]
Warm-Up #3

Compare the combination array below with Pascal’s Triangle.

\[
\begin{array}{cccccccc}
0 & 0 & & & & & & 1 \\
1 & 1 & & & & & & 1 \\
2 & 1 & & & & & & 1 \\
3 & 2 & 1 & & & & & 1 \\
4 & 3 & 3 & 1 & & & & 1 \\
5 & 4 & 6 & 4 & 1 & & & 1 \\
6 & 5 & 10 & 10 & 5 & 1 & & 1 \\
\end{array}
\]

a) Complete the next row in both tables.

b) Pascal derived the following theorem known as Pascal’s Theorem.

Verify the theorem for \( n = 5 \) and \( r = 1 \).

Warm-Up #4

The Sum of the Numbers in the \( k^{th} \) Row of Pascal’s Triangle.

Complete the table which gives the sum of the numbers in each of the first six rows of Pascal’s Triangle.

<table>
<thead>
<tr>
<th>Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) Complete the following statement:

“The sum of the numbers in the \( k^{th} \) row of Pascal’s Triangle is _______.”

b) Look at row 7 in the diagram at the top of the page. This row has a sum of 26.

Using the above rule we can see that

\[ 6 \binom{0}{0} + 6 \binom{1}{1} + 6 \binom{2}{2} + 6 \binom{3}{3} + 6 \binom{4}{4} + 6 \binom{5}{5} + 6 \binom{6}{6} = 2^6 \]

In general,

\[ \sum_{r=0}^{n} \binom{n}{r} = 2^n \]
Warm-Up #5

The Sum of the Coefficients in the Expansion of \((a + b)^n\)

Consider the expansion of \((a + b)^n\).

\[
(a + b)^1 = a + b \\
(a + b)^2 = a^2 + 2ab + b^2 \\
(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\]

Complete the table which gives the sum of the coefficients in each of the first five expansions of \((a + b)^n\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sum of coefficients</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Complete the following statement:

“The sum of the coefficients in the expansion of \((a + b)^n\) is _______."

Memorize these results.

- The sum of the numbers in the \(k\)th row of Pascal’s Triangle is \(2^k - 1\)
- \(nC_0 + nC_1 + nC_2 + nC_3 + ... + nC_{n-1} + nC_n = 2^n\)
- The sum of the coefficients in the expansion of \((a + b)^n\) is \(2^n\)

Class Ex. #1

a) What is the sum of the numbers in the tenth row of Pascal’s Triangle?

b) What is the sum of the coefficients in the expansion of \((m + n)^{12}\)?

c) What is the sum of: i) \(15C_0 + 15C_1 + 15C_2 + ... + 15C_{15}\)?

ii) \(6C_0 + 6C_1 + 6C_2 + ... + 6C_5\)?

Complete Assignment Questions #1 - #6
Warm-Up #6  Discovering the Binomial Theorem

Referring to the combination array and Pascal’s Triangle in Warm-Up #3, it is not just a coincidence that the combination symbols represent the numbers in Pascal’s Triangle.

To investigate this consider the expansion of \((x + y)^6\).

\[
(a + b)^6 = (a + b)(a + b)(a + b)(a + b)(a + b)(a + b)
\]

\[
= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\]

Notice that each term in the expansion is a combination of \(a\’s\) and \(b\’s\).

Consider how the third term (i.e.\(15a^4b^2\)) is formed. It is formed by choosing \(b\) from any two of the six factors in the expansion of \((a + b)^6\) and an \(a\) from the remaining four factors. The two \(b\’s\) can be chosen in \(\binom{6}{2}\) (or 15) ways. The four \(a\’s\) can then be chosen in \(\binom{4}{4}\) (or 1) way. This leads to the term \(\binom{6}{2}a^4b^2\).

• Write the coefficients in terms of combinations for the expansion below.

\[
(a + b)^6 = (a + b)(a + b)(a + b)(a + b)(a + b)(a + b)
\]

\[
= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\]

\[
= \binom{6}{2}a^4b^2
\]

This leads to the following general formula for binomial expansions known as the Binomial Theorem.

Binomial Theorem

\[
(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n}b^n, \text{ where } n \in \mathbb{I}, n \geq 0
\]

General Term of the Expansion of \((a + b)^n\)

The term \(\binom{n}{k}a^{n-k}b^k\) is called the general term of the expansion. It is the \((k + 1)^{th}\) term in the expansion (not term \(k\))

\[
t_{k+1} = \binom{n}{k}a^{n-k}b^k
\]

This formula is on the formula sheet.
Class Ex. #2

Find:  
\( a) \) the fifth term of \((a + b)^8\)  \( b) \) the middle term of \((2a - 5)^6\).

Class Ex. #3

Expand \((3x - 2)^3\).

Class Ex. #4

One term in the expansion of \((x + a)^{10}\) is \(3281250x^4\). Determine the numerical value of \(a\).

Class Ex. #5

Find the constant term (i.e. the term independent of \(x\)) in the expansion of \(\left(2x - \frac{1}{x^2}\right)^{15}\).

Complete Assignment Questions #7 - #13
Assignment

1. Consider the expansion of \((a + b)^n\).
   a) What are the coefficients of the first and last terms?
   b) How many terms are there in the expansion?
   c) What is the sum of the exponents in each term?
   d) What is the coefficient of the third term?
   e) The term in which position contains \(b^4\)?

2. What are the first three terms in the ninth row of Pascal’s Triangle?

3. What are the last three terms in the sixteenth row of Pascal’s Triangle?

4. Find the value of \(n\) if the expansion of:
   a) \((2x + 3)^n\) has 18 terms
   b) \((3x - 5)^{4n - 3}\) has 26 terms.

5. a) What is the sum of the terms in the ninth row of Pascal’s Triangle?
   b) What is the sum of the coefficients in the expansion of \((a + b)^9\)?
   c) What is the sum \(9C_0 + 9C_1 + 9C_2 + \ldots + 9C_9\)?

6. Find the following sums:
   a) \(10C_1 + 10C_2 + 10C_3 + \ldots + 10C_{10}\)
   b) \(7C_6 + 7C_5 + 7C_4 + \ldots + 7C_1\)

7. Find the indicated term of each expansion.
   a) the fifth term of \((a - b)^5\)
   b) the second term of \((x - 2)^6\)
c) the third term of \((3x + 2y)^9\)  
d) the fourth term of \((a^2 - 2a)^7\)

e) the middle term of \(\left(2 - \frac{x}{2}\right)^6\)

8. Expand and write in the simplest form:

a) \((2a + b)^3\)

b) \(\left(x - \frac{1}{x}\right)^4\)

9. Find the indicated term of each expansion.

a) the term in \(x^3\) in \((1 - 2x)^{12}\)  
b) the term in \(x^4y^3\) in \((3x - y)^7\)

10. a) One term in the expansion of \((x + a)^8\) is \(448x^6\).  
    Determine the value of \(a\), \(a > 0\).  
b) One term in the expansion of \((x + b)^{11}\) is \(-4455x^8\).  
    Determine the value of \(b\).
11. Find the indicated term of each expansion.
   a) the constant term in the expansion of \( \left( x^2 - \frac{1}{x} \right)^6 \)
   b) the term independent of \( x \) in \( \left( 2x + \frac{1}{x^3} \right)^8 \)

12. If \((2a + 3)^{2n+3}\) has 18 terms, then the value of \( n \) is _____.

13. The term in \( x^{11} \) in the expansion of \( \left( x^2 + \frac{1}{x} \right)^{10} \) has a numerical coefficient, to the nearest whole number, of _____.

**Answer Key**

1. a) 1       b) \( n + 1 \)       c) \( n \)       d) \( \binom{n}{2} = \frac{n(n-1)}{2} \)       e) term 5

2. 1, 8, 28       3. 105, 15, 1       4. a) 17       b) 7

5. a) 256       b) 512       c) 512       6. a) 1023       b) 126

7. a) \( 5ab^4 \)       b) \(-12x^5 \)       c) \( 314 928 x^7 y^2 \)       d) \(-280a^{11} \)       e) \(-20x^3 \)

8. a) \( 8a^3 + 12a^2b + 6ab^2 + b^3 \)       b) \( x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4} \)

9. a) \(-1760x^3 \)       b) \(-2835x^4y^3 \)

10. a) 4       b) \(-3 \)

11. a) 15       b) 1792

12. 7       13. 120

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Permutations and Combinations Lesson #7: Pathway Problems

Warm-Up

Consider the following problem:

“Find the number of pathways from A to B if paths must always move closer to B”

This problem can be solved in several ways.

a) Solve this problem by tracing the number of paths on the grid.

b) Explain how this problem can be regarded as an example of permutations with repetitions. Determine the number of pathways using this approach.

c) Explain how this problem can be regarded as an example of combinations. Use the combination formula or Pascal’s Triangle to determine the number of pathways.

Class Ex. #1

A city centre has a rectangular road system with 5 streets running north to south and 6 avenues running west to east.

a) Draw a grid to represent this situation.

b) Sean is driving a car and is situated at the extreme northwest corner of the city centre. In how many ways can he drive to the extreme southeast corner if at each turn he moves closer to his destination (assume all streets and avenues allow two-way traffic).
Find the number of pathways from $A$ to $B$ if paths must always move closer to $B$.

A taxi company is trying to find the quickest route during rush hour traffic from the train station to the football stadium. How many different routes must be considered if at each intersection the taxi must always move closer to the football stadium?

How many different paths can a pinball take as it falls from top to bottom?

Complete Assignment Questions #1 - #7
Assignment

1. Find the number of pathways from A to B if paths must always move closer to B.

   a)

   b)

   c)

   d)

   e)

   f)

   g)

   h)

   i)

   j)

   k)
2. The bakery is four blocks south and three blocks west of the supermarket. The bakery driver, bored with travelling the same route, decides to use a different route for each delivery. Assuming that he always travels closer to the supermarket, how many deliveries are possible before he has to repeat a route?

3. A town has 10 streets running from north to south and 8 avenues running from west to east. A man wishes to drive from the extreme south-west intersection to the extreme north-east intersection, moving either north or east along one of the streets or avenues. Find the number of routes he can take. State any assumptions that you have made.

4. Given the following pinball machines determine the number of pathways to reach each of the exits.

   a)
   
   b)

5. The number of pathways from $X$ to $Y$ if paths must always move closer to $Y$ is

   A. $\frac{6!}{3! \ 3!} + \frac{4!}{2! \ 2!}$
   
   B. $\frac{6!}{3! \ 3!} \times \frac{4!}{2! \ 2!}$
   
   C. $\frac{8!}{4! \ 4!} + \frac{6!}{3! \ 3!}$
   
   D. $\frac{8!}{4! \ 4!} \times \frac{6!}{3! \ 3!}$
6. The number of different paths that a pinball can take as it falls from top to bottom is ____.

7. Counsellor McAdam has to drive from city hall (at the corner of 1st Ave. and 1st St.) to the hospital (at the corner of 6th Ave. and 6th St.). Third Avenue is one way heading east and Fourth Avenue is one way heading west. If he must always be travelling closer to the hospital, the number of different routes he can take is _____.

Answer Key

1. a) 126  b) 60  c) 106  d) 140  e) 90  f) 900  g) 260  h) 495  i) 59  j) 66  k) 23
2. 35 3. 11 440. Assumptions: 1) He always travels closer to his destination.
   2) None of the routes are one-way from north to south or east to west
4. a) Exit A: 1  Exit B: 5  Exit C: 10  Exit D: 10  Exit E: 5  Exit F: 1
   b) Exit A: 20  Exit B: 15  Exit C: 6
5. B 6. 53 7. 126
Probability Lesson 1: 
Probability Terminology and Notation

Warm-Up

The following simple probability questions were asked in Junior High Math.

a) A fair die is rolled. What is the probability of rolling a “one”?

b) A circular spinner is divided into four equal sectors, labelled clubs, diamonds, hearts and spades. When it is spun what is the probability it lands on “hearts”?

c) Two coins are thrown and the number of heads is counted. What is the probability of obtaining “two heads”?

Do not calculate the probabilities at this time.

Terminology

Probability theory deals with the mathematics of chance or prediction. The following terminology will be used:

- A **trial** is any operation whose outcome cannot be predicted with certainty.
  eg. a coin is tossed

- An **experiment** consists of one or more trials.
  eg. a coin is tossed, three dice are rolled.

- An **outcome** is the result of carrying out an experiment.
  eg. H, 6 6 4,

- The **sample space** \( S \) of an experiment is the set of all possible outcomes.
  eg. \{H, T\}, \{1, 2, 3, 4, 5, 6\}, \{(1,1),(1,2), \ldots (6,6)\}

- An **event** is a subset of the sample space. It consists of one or more of the possible outcomes of an experiment.
  eg. \{H\}, \{an even number\}

- If an experiment has a set of **equally likely** possible outcomes then the probability of a particular event \( A \) is given by the formula:

\[
P(A) = \frac{\text{number of outcomes favourable to } A}{\text{total number of possible outcomes}}
\]

eg. \( P(H) = \frac{1}{2} \)

- If the event \( X \) does not include any of the outcomes in the sample space, then the event \( X \) is **impossible** and we write \( P(X) = 0 \).
  eg. \( P(\text{rolling a 9 on a standard die}) = 0 \)

- If the event \( Y \) includes all of the outcomes in the sample space, then the event \( Y \) is **certain** and \( P(Y) = 1 \).
  eg. \( P(\text{rolling a natural number less than 7 on a standard die}) = 1 \)
• For any event \( A \), \( 0 \leq P(A) \leq 1 \).

• The notation \( P(\overline{A}) \) means the probability of “not \( A \),” the complement of \( A \), i.e. \( P(\text{not } A) = P(\overline{A}) = 1 - P(A) \).

  eg. if \( P(\text{rain tomorrow}) = \frac{1}{3} \), then \( P(\text{no rain tomorrow}) = 1 - \frac{1}{3} = \frac{2}{3} \).

  The formula \( P(\overline{A}) = 1 - P(A) \) is on the formula sheet.

---

**Class Ex. #1**

a) For each of the examples in the *Warm Up* state:
   i) the sample space
   ii) the outcomes in the sample space which are favourable to the event indicated
   iii) whether the outcomes are equally likely
   iv) the probability of the event

• Warm-Up #1a)
  A fair die is rolled. What is the probability of rolling a “one”? 

• Warm-Up #1b):
  A circular spinner is divided into four equal sectors, labelled clubs, diamonds, hearts and spades. When it is spun what is the probability it lands on “hearts”?

• Warm-Up #1c):
  Two coins are thrown and the number of heads is counted. What is the probability of obtaining “two heads”?

b) Why can you not directly use the formula \( P(A) = \frac{\text{number of outcomes favourable to } A}{\text{total number of possible outcomes}} \) to calculate the probability in Warm-Up #1c)?

c) Explain how you calculated the probability.
**Compound Events**

Events formed from repeated trials or from a combination of simple events are called **compound events** and often a table, a chart or a tree diagram is useful in determining the sample space.

Consider an experiment of rolling an equally spaced triangular spinner numbered 1 to 3 and tossing two coins.

a) Draw a tree diagram to show all the outcomes for the experiment. List the sample space at the end of the tree.

b) How many elements are in the sample space? How can you use the fundamental counting principle to determine the answer?

c) Are all the outcomes equally likely?

d) State the probability of obtaining:
   i) a three and two heads
   ii) a prime number and exactly one tail

A blue die and a red die are rolled. The outcome “3 on the blue die and 4 on the red die” can be represented by the ordered pair (3, 4).

a) Show all the possible outcomes in the array.

b) How many points are in the sample space?

c) List the event “the same number appears on both dice” as a subset of the sample space.

d) State the probabilities of the following events:
   i) the same number appears on both dice
   ii) a different number appears on each die

---

### Complete Assignment Questions #1 - #11
Assignment

1. A machine produces batteries. As part of a quality control process, six batteries are randomly selected every hour and tested. The number of defective batteries is noted.
   a) List the sample space.
   b) List the following events as subsets of the sample space:
      i) at least two batteries are defective   ii) at most two batteries are defective
   c) List the complement of the following events as subsets of the sample space:
      i) three batteries are defective   ii) at most five batteries are defective

2. Eight coins are tossed and the number of tails is counted.
   a) List the sample space.
   b) List the event “more tails than heads” in terms of the sample space.
   c) List the event “more heads than tails” in terms of the sample space.
   d) List the complement of the following events as subsets of the sample space:
      i) an even number of tails   ii) at least one tail

3. A committee of two people is to be selected from; Alex (A), Bob(B), Carol(C), Dave(D), and Ed(E).
   a) List the sample space.
   b) List the following events as subsets of the sample space:
      i) Alex is on the committee   ii) neither Carol nor Dave is on the committee
   c) Assuming each element of the sample space to be equally likely, determine
      i) \(P(\text{Alex is on the committee})\)   ii) \(P(\text{Bob and Dave are on the committee})\)
      iii) \(P(\text{Alex is not on the committee})\)
4. Packet 1 contains an orange candy, a lemon candy and a strawberry candy. Packet 2 contains an orange candy, a strawberry candy and a raspberry candy. Without looking in the packets a student takes one candy from packet 1, then one candy from packet 2.

a) Use a tree diagram to determine the sample space for this experiment?

b) List the event “both candies are the same type” as a subset of the sample space.

c) List the event “a strawberry candy is not chosen” as a subset of the sample space.

d) State the probability of the following events:
   i) both candies are the same
   ii) a strawberry candy is not chosen
   iii) at least one orange candy is chosen
   iv) the candies are different types

e) Explain the relationship between the answers to parts i) and iv)

5. Four dimes are tossed by Student 1 and five quarters are tossed by Student 2. The number of heads is recorded by each student.

a) List the sample space as an array of ordered pairs

b) List the event “Student 1 recorded more heads than Student 2” in terms of the sample space.

c) State the probability of the following events:
   i) Student 1 recorded more heads than Student 2.
   ii) Student 1 recorded less heads than Student 2.

d) Is the event in c)ii) the complement of the event in c)i)? Explain.
6. Rebecca, Elizabeth, and Jenny each toss a coin to see who is going to pay for coffee.

- If one of the students throws a different outcome from the other two then that student buys all three coffees.
- If all the students throw the same outcome then each student pays for their own coffee.

a) Use a tree diagram to construct a sample space for tossing three coins.

b) State the probabilities of the following events:
   i) Rebecca buys all the coffees.
   ii) each student buys their own
   iii) Jenny gets a free coffee.

7. An experiment consists of rolling a die, flipping a coin, and spinning a spinner divided into 5 regions. The number of elements in the sample space of this experiment is

   A. 3
   B. 7
   C. 13
   D. 60

8. If \( P(X) = 0.2 \), then \( P(\overline{X}) \) equals

   A. \(-0.2\)
   B. 0.2
   C. 0.5
   D. 0.8
Box A contains a red marble, a blue marble and a yellow marble. Box B contains a red marble, a blue marble, a yellow marble and a green marble. One marble is taken from each box. The probability, to the nearest hundredth, that both marbles are the same colour is ______.

Use the following information to answer questions #10 - #11

Samantha spins a pointer at a carnival similar to the one shown. If the pointer lands on A she will win a medium sized stuffed rabbit. If the pointer lands on B she will a small stuffed teddy bear. If the pointer lands on C she will win a large stuffed unicorn.

10. The probability, to three decimal places, that Samantha wins a stuffed rabbit is ______.

11. The probability, to the nearest hundredth, that Samantha does not win a unicorn is ______.

Answer Key

1. a) \{0, 1, 2, 3, 4, 5, 6\} b) i) \{2, 3, 4, 5, 6\} ii) \{0, 1, 2\} c) i) \{0, 1, 2, 4, 5, 6\} ii) \{6\}

2. a) \{0, 1, 2, 3, 4, 5, 6, 7, 8\} b) \{5, 6, 7, 8\} c) \{0, 1, 2, 3\} d) i) \{1, 3, 5, 7\} ii) \{0\}

3. a) \{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE\} b) i) \{AB, AC, AD, AE\} ii) \{AB, AE, BE\} c) i) \frac{2}{5} ii) \frac{1}{10} iii) \frac{3}{5}

4. a) \{OO, OS, OR, LO, LS, LR, SO, SS, SR\} b) \{OO, SS\} c) \{OO, OR, LO, LR\} d) i) \frac{2}{9} ii) \frac{4}{9} iii) \frac{5}{9} iv) \frac{7}{9}

5. a) 30 pairs from (0, 0) to (4, 5) b) \{(1,0), (2,0), (3,0), (4,0), (2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\}

c) i) \frac{1}{3} ii) \frac{1}{2} d) No, because there are outcomes where # heads tossed by each student is the same

6. a) \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} b) i) \frac{1}{4} ii) \frac{1}{4} iii) \frac{1}{2}

7. D 8. D 9. 0.25 10. 0.375 11. 0.875
Probability Lesson #2:  
**Mutually Exclusive Events and the Event “A or B”**

**The Events “A or B”, “A and B”**

In mathematics the event \( A \) or \( B \) is said to occur if the event \( A \) occurs or if the event \( B \) occurs or if both events occur.

The event \( A \) and \( B \) occurs if both event \( A \) and event \( B \) occur simultaneously.

**Warm-Up #1**

Consider the experiment of rolling a die and noting the result. Let the event \( A \) be “an even number is thrown” and the event \( B \) be “an odd number is thrown”.

a) Mark the outcomes to the experiment on the Venn Diagram which represents the sample space.

b) List the outcomes for:

i) the event \( A \)

ii) the event \( B \)

iii) the event \( A \) or \( B \)

iv) the event \( A \) and \( B \)

c) Let \( n(A) \) represent the number of outcomes in event \( A \).

Complete the following:

\[ n(A) = \quad n(B) = \quad n(A \text{ or } B) = \quad n(A \text{ and } B) = \]

d) Determine the following probabilities:

\[ P(A) = \quad P(B) = \quad P(A \text{ or } B) = \quad P(A \text{ and } B) = \]

- In this experiment the events \( A, B \) have no common outcomes.
  The events \( A, B \) are called **mutually exclusive**.

- Notice that in the Venn Diagram the circle for \( A \) and the circle for \( B \) have no area of overlap.

e) Verify the following rule.

\[ P(A \text{ or } B) = P(A) + P(B) \]
Warm-Up #2

Consider the experiment of rolling a die and noting the result. Let the event $A$ be “an even number is thrown” and the event $B$ be “a multiple of three” is thrown.

a) Mark the outcomes to the experiment on the Venn Diagram.

b) List the outcomes for:
   i) the event $A$
   ii) the event $B$
   iii) the event $A$ or $B$
   iv) the event $A$ and $B$

Complete the following:

\[ n(A) = \quad n(B) = \quad n(A \text{ or } B) = \quad n(A \text{ and } B) = \]

d) Determine the following probabilities:

\[ P(A) = \quad P(B) = \quad P(A \text{ or } B) = \quad P(A \text{ and } B) = \]

- In this experiment the events $A$, $B$ have common outcomes. The events $A$, $B$ are not mutually exclusive.

- Notice that in the Venn Diagram the circle for $A$ and the circle for $B$ have an area of overlap representing the event $A$ and $B$.

e) Verify the following rule.

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

Mutually Exclusive Events

Events are said to be mutually exclusive if they have no common outcomes.

Probability of the Event $A$ or $B$

Use the following formulas for the probability of the event $A$ or $B$.

\[
\text{If the events } A, B, \text{ are mutually exclusive then} \\
P(A \text{ or } B) = P(A) + P(B)
\]

This formula is NOT on the formula sheet

\[
\text{If the events } A, B, \text{ are NOT mutually exclusive, then} \\
P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)
\]

This formula is on the formula sheet
In each case, state whether the events $A, B$ are mutually exclusive or not.

a) Experiment - a card is drawn from a standard deck
   
   Event $A$ - a face card is selected  
   Event $B$ - a club is selected

b) Experiment - two dice are thrown
   
   Event $A$ - the dice both show the same value  
   Event $B$ - the total score is 11

Consider the experiment of drawing a card from a standard deck. The following events are defined:

- Event $A$ - a face card is selected
- Event $B$ - a club is selected
- Event $C$ - an ace is selected
- Event $D$ - a red card is selected

State all the pairs of events which are mutually exclusive.

Use the Venn Diagram to state all the pairs of events which are mutually exclusive.

Use the following information to determine whether the events $A, B$ are mutually exclusive.

$$P(A) = \frac{1}{4} \quad P(B) = \frac{1}{3} \quad P(A \text{ or } B) = \frac{7}{12}$$

A grade 9 class were surveyed to find out whether they did Math homework or English homework last night. The Venn diagram shows the percentage of students in each category.

If a student is selected at random from the class determine the probability that the student last night did:

a) Math homework  

b) Math and English homework  

c) Math or English homework
A card is drawn from a standard deck of 52 cards. Use formulas to determine the probability that:

a) A nine of diamonds or a heart is drawn.

b) A nine or a heart is drawn.

200 people with neurology symptoms, which include headaches and backaches, participated in a study to evaluate a pain relief medicine. All the people took the medicine and the results were as follows:

- 60 people experienced headache relief
- 126 people experienced backache relief
- 36 people experienced relief from both

What is the probability that a person who takes the drug experiences relief from

a) at least one of the two symptoms?

b) neither of the symptoms?

---

**Assignment**

1. In each case, state whether the events $A$, $B$ are mutually exclusive or not.

   a) Experiment - a card is drawn from a standard deck
      
      Event $A$ - a spade is selected  
      Event $B$ - a club is selected

   b) Experiment - a card is drawn from a standard deck
      
      Event $A$ - a spade is selected  
      Event $B$ - a black seven is selected

   c) Experiment - a card is drawn from a standard deck
      
      Event $A$ - a red two is selected  
      Event $B$ - a red face card is selected

   d) Experiment - two dice are thrown
      
      Event $A$ - the dice both show odd numbers  
      Event $B$ - the total score is 8

   e) Experiment - a chocolate bar is chosen
      
      Event $A$ - the bar contains raisins  
      Event $B$ - the bar contains nuts

   f) Experiment - a student is chosen from this class
      
      Event $A$ - the student is left handed  
      Event $B$ - the student is female
2. Use the Venn Diagram to state all the pairs of events which are mutually exclusive.
   a) 
   ![Venn Diagram](image)
   b) 
   ![Venn Diagram](image)

3. Meghan (M), Jolene (J), and Tara (T) are the only entrants in a competition for which there is a first and second prize. Consider the experiment of selecting the two prize winners.
   a) List the sample space.
   b) State as subsets of the sample space the events
      i) \( P \) – Meghan wins first prize
      ii) \( Q \) – Meghan wins second prize
      iii) \( R \) – Meghan wins a prize
      iv) \( S \) – Jolene and Tara win the prizes
   c) Which three of these events are mutually exclusive?
   d) List all pairs of these events that are mutually exclusive.

4. Use the following information to determine whether the events \( A, B \) are mutually exclusive.
   a) \( P(A) = \frac{1}{2}, \ P(B) = \frac{1}{5}, \ P(A \ and \ B) = 0 \)
   b) \( P(A) = 0.3, \ P(B) = 0.2, \ P(A \ or \ B) = 0.4 \)
   c) \( P(A) = \frac{2}{5}, \ P(B) = \frac{1}{4}, \ P(A \ or \ B) = \frac{13}{20} \)
5. Use the given information to calculate the probability of the indicated event.
   a) Tanya and Fujiko are two of the competitors in the final of the long jump competition. The probability that Tanya wins the competition is \( \frac{2}{7} \), and the probability that Fujiko wins is \( \frac{1}{5} \). Calculate the probability that Tanya or Fujiko win the competition.

   b) In a group of students, the probability that a student chosen at random walks to school is 0.35, and the probability that a student has blonde hair is 0.2. If the probability that a student walks to school or has blonde hair is 0.45, calculate the probability that a student walks to school and has brown hair.

   c) Mohinder is playing with an incomplete deck of cards. When one card is selected at random from the incomplete deck, the probability that it is a heart is 0.3, and the probability that it is a face card is 0.2. If the probability that it is a heart and a face card is 0.1, calculate the probability that it is a heart or face card.

6. A card is drawn from a standard deck. Use an appropriate formula to determine the probability that:
   a) a face card or a club is drawn
   b) a red card or a club is drawn.
   c) an eight or a spade is drawn
   d) a five of diamonds or a Queen is drawn
   e) a black card or a diamond is drawn
   f) a Jack or a red card is drawn.
7. Allyson and Brittney were solving the following problem:

100 people in a community participated in a technology survey. It was found that 80 people have a computer, 40 people have access to the internet, and 30 people have both a computer and access to the internet. If one of these people is chosen at random, what is the probability that the person has

a) a computer or access to the internet?  

b) neither?

Allyson chose to do the problem using a Venn diagram. Brittney chose to use probability formulas.

Show each student’s method of solution below:

<table>
<thead>
<tr>
<th>Allyson’s Solution</th>
<th>Brittney’s Solution</th>
</tr>
</thead>
</table>

8. Mr. Spark has problems starting his two cars during the cold winter months. He decided to record the number of times each car starts during a one month period in the winter. He attempted to start each car every morning for thirty days. He recorded the following information.

- His first car started 20 times
- His second car started 18 times
- Both cars started 40% of the time

What is the probability that on any particular morning during the month

a) at least one of the cars start?  

b) he cannot start either of his two cars?
9. In a group of 80 elementary school children, 48 children liked cornflakes, 22 children liked both cornflakes and oatmeal, and 20 children liked neither cornflakes nor oatmeal. Calculate the probability that a child selected at random from the group liked only oatmeal.

10. In a school, 55% of the students have passed their English examination and 45% have passed their Mathematics examination. Comment critically on the following statement.

“The probability the student chosen at random has passed his or her English examination or Mathematics examination is 0.55 + 0.45 = 1.”

Use the following information to answer questions #11 and #12.

In an election there are three candidates. Candidate A is twice as likely to win as candidate B and candidate B is twice as likely to win as candidate C.

11. The probability that candidate A wins the election is

A. $\frac{2}{3}$  B. $\frac{3}{4}$  C. $\frac{4}{5}$  D. $\frac{4}{7}$

12. The probability, expressed as a decimal to the nearest hundredth, that candidate A or candidate B wins the election is _____.

Answer Key

1. a) yes  b) no  c) yes  d) no  e) no  f) probably no (depends on the class)
3. a) $\{MJ, MT, JM, JT, TM, TJ\}$  b) i) $\{MJ, MT\}$ ii) $\{JM, TM\}$ iii) $\{MJ, MT, JM, TM\}$ iv) $\{JT, TJ\}$
4. a) Yes, since $P(A \text{ and } B) = 0$  b) No, since $P(A) + P(B) \neq P(A \text{ or } B)$
5. a) $\frac{17}{35}$  b) 0.1  c) 0.4
6. a) $\frac{11}{26}$  b) $\frac{3}{4}$  c) $\frac{4}{13}$  d) $\frac{5}{32}$  e) $\frac{3}{4}$  f) $\frac{7}{13}$
7. a) $\frac{9}{10}$  b) $\frac{1}{10}$  8. a) $\frac{13}{15}$  b) $\frac{2}{15}$
9. $\frac{3}{20}$
10. The two events are not mutually exclusive, so the statement is false
11. D  12. 0.86
Probability Lesson #3:  
Conditional Probability and the Event “A and B”

**Warm-Up #1**

One card is drawn from a deck of cards and is not replaced. A second card is then drawn. Consider the following events  
A = \{ the first card is a heart \}  
B = \{ the second card is a heart \}  
a) Determine $P(A)$  
b) Why is it not possible to determine $P(B)$?

The probability of event $B$ depends on whether or not event $A$ occurred. Events $A$ and $B$ are called **dependent events**.

**Dependent Events**

Two events are **dependent** if the knowledge that one event has occurred changes the probability of the other event occurring.

**Warm-Up #2**

One card is drawn from a deck of cards and is replaced. A second card is then drawn. Consider the following events  
A = \{ the first card is a heart \}  
B = \{ the second card is a heart \}  
a) Determine $P(A)$  
b) Determine $P(B)$

The probability of event $B$ does NOT depend on whether or not event $A$ occurred. Events $A$ and $B$ are called **independent events**.

**Independent Events**

Two events are **independent** if the knowledge that one event has occurred has no effect on the probability of the other event occurring.

Classify the following events as dependent or independent.

a) The experiment is rolling a die and tossing a coin.  
The first event is rolling 2 on the die and the second event is tossing tails on the coin.

b) The experiment is choosing two cards without replacement from a standard deck.  
The first event is that the first card is a king and the second event is that the second card is a king.

c) The experiment is choosing two cards with replacement from a standard deck.  
The first event is that the first card is a king and the second event is that the second card is a king.
Conditional Probability

Consider the scenario from Warm-Up #1.
One card is drawn from a deck of cards and is not replaced. A second card is then drawn.
Consider the following events.
\[ A = \{ \text{the first card is a heart} \} \quad B = \{ \text{the second card is a heart} \} \]

\[ P(A) = \frac{13}{52} = \frac{1}{4} \quad \text{but } P(B) \text{ depends on whether the event } A \text{ occurred or did not occur.} \]

- We denote by \( P(B|A) \) the conditional probability of the event \( B \), given that the event \( A \) has occurred.
  In this example \( P(B|A) = \) ______

- We denote by \( P(B|\overline{A}) \) the conditional probability of the event \( B \), given that the event \( A \) has NOT occurred.
  In this example \( P(B|\overline{A}) = \) ______

Warm-Up #3

A red die and a blue die are tossed. The outcomes are shown in the array.
Consider the following two events:
Event \( A \) \{the sum of the two dice is 10\}
Event \( B \) \{the number on each die is the same\}
a) List the outcomes for the following events as subsets of the sample space.
  i) \( A = \) \_
  ii) \( B = \) \_
  iii) \( A \) and \( B = \) \_

b) State the following probabilities:
  i) \( P(A) = \) \_
  ii) \( P(B) = \) \_
  iii) \( P(B|A) = \) \_
  iv) \( P(A|B) = \) \_
  v) \( P(A \text{ and } B) = \) \_
  vi) \( P(B \text{ and } A) = \) \_

c) Verify the following:
  i) \( P(A \text{ and } B) = P(A) \times P(B|A) \)  \quad \text{ii) } P(A \text{ and } B) = P(B) \times P(A|B) \)
**Multiplication Law for Dependent Events**

Given that two events \(A, B\), are dependent, then

\[
P(A \text{ and } B) = P(A) \times P(B \mid A)
\]

*This formula is on the formula sheet*

- The notation \(P(B/A)\) is sometimes used instead of \(P(B \mid A)\)

**Class Ex. #2**

Two cards are drawn without replacement from a standard deck of 52 cards. Determine the probability of the following events:

a) both cards are red  
b) neither card is a club

c) the first card is a king and the second card is a five.  
d) one of the cards is a king and the other is a five.

---

**Multiplication Law for Independent Events**

If the events \(A, B\), are independent, then the knowledge that event \(A\) has occurred has no effect on the probability of the event \(B\) occurring.

This means that \(P(B \mid A) = P(B)\).

Therefore we have the following law for independent events:

Given that two events \(A, B\), are independent, then

\[
P(A \text{ and } B) = P(A) \times P(B)
\]

*This formula is NOT on the formula sheet*

**Class Ex. #3**

Two cards are drawn with replacement from a standard deck of 52 cards. Determine the probability of the following events:

a) both cards are red  
b) the first card is a king and the second card is a five.

c) one of the cards is a king and the other is a five.
Often students confuse the concept of **mutually exclusive** events with that of **independent** events. These terms do NOT mean the same thing.

- The concept of **mutually exclusive events**, involves whether or not two events can occur simultaneously.
- The concept of **independent events** involves whether or not the occurrence of one event has an effect on the probability of the other event occurring.

**Class Ex. #4**

If \( P(A) = \frac{1}{3} \), \( P(B) = \frac{2}{5} \) and \( P(A \text{ or } B) = \frac{3}{5} \), investigate whether the events \( A \) and \( B \) are:

a) mutually exclusive events  

b) independent events

**Class Ex. #5**

Let \( A \) and \( B \) be events with \( P(A) = \frac{1}{2} \), \( P(B) = \frac{1}{3} \), and \( P(A \text{ and } B) = \frac{1}{4} \). Find:

a) \( P(A \mid B) \)  

b) \( P(B \mid A) \)  

c) \( P(A \text{ or } B) \)

**Class Ex. #6**

The probability that Ashley will pass Math this semester is 0.7 and the probability that she will pass English this semester is 0.9. If these events are independent, what is the probability (to the nearest hundredth) that she will pass

a) Math and English  

b) Math or English  

c) Math but not English  

d) neither Math nor English

Complete Assignment Questions #1 - #14
Assignment

1. Classify the following events as dependent or independent.

   a) The experiment is to consider the height and weight of students.
      The first event is that the student is greater than 1.8 m tall and the second event is that
      the student weighs more than 70 kg.

   b) The experiment is choosing two cards with replacement from a standard deck.
      The first event is that the first card is a jack and the second event is that the second card
      is a queen.

   c) The experiment is choosing two cards without replacement from a standard deck.
      The first event is that the first card is a seven and the second event is that the second
      card is a seven.

   d) The experiment is rolling a die and rolling the die again.
      The first event is that the number on the first roll is a six and the second event is that the
      number on the second roll is a two.

2. Consider two events such that \( P(A) = \frac{1}{2} \), \( P(B) = \frac{1}{3} \), and \( P(A \text{ or } B) = \frac{3}{4} \).
   a) Are \( A, B \) mutually exclusive events?
   b) Are \( A, B \) independent events?

3. Consider two events such that \( P(A) = \frac{1}{4} \), \( P(B) = \frac{2}{5} \), and \( P(A \text{ and } B) = \frac{1}{10} \).
   a) Are \( A, B \) mutually exclusive events?
   b) Are \( A, B \) independent events?
4. Let \( A \) and \( B \) be events with \( P(A) = 0.6 \), \( P(B) = 0.4 \), and \( P(A \text{ and } B) = 0.3 \). Find:
   a) \( P(A \mid B) \)  
   b) \( P(B \mid A) \)  
   c) \( P(A \text{ or } B) \)

5. Let \( A \) and \( B \) be events with \( P(A) = \frac{3}{8} \), \( P(B) = \frac{5}{8} \), and \( P(A \text{ or } B) = \frac{3}{4} \). Find:
   a) \( P(A \mid B) \)  
   b) \( P(B \mid A) \)

6. A red die and a blue die are tossed. What is the probability that the red die shows a 1 and the blue die shows a 5 or a 6?

7. Two cards are drawn with replacement from a standard deck of 52 cards. Determine the probability of the following events:
   a) both cards are spades  
   b) both cards are sevens  
   c) neither card is red  
   d) the first card is a club and the second card is a diamond.  
   e) one of the cards is red and the other is black.
8. Two cards are drawn without replacement from a standard deck of 52 cards. Determine the probability of the following events:
   a) both cards are spades   b) both cards are sevens   c) neither card is red
   d) the first card is a club and the second card is a diamond.
   e) one of the cards is red and the other is black.

9. The probabilities that Sara will pass grade 12 math and grade 12 physics this semester are 0.85 and 0.75 respectively. If these events are independent, what is the probability (to four decimal places) that she will pass:
   a) both math and physics   b) math but not physics
   c) physics but not math   d) neither math nor physics

10. In a city school, 60% of students have blue eyes, 55% have dark hair, and 20% have neither blue eyes nor dark hair.
    a) Find the probability that a randomly selected student will have blue eyes and dark hair.
    b) State with a reason if these two characteristics are independent.

11. Consider two events such that $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \text{ or } B) = 0.58$.
    The events $A, B$ are
    A. Mutually exclusive and dependent
    B. Not mutually exclusive and dependent
    C. Mutually exclusive and independent
    D. Not mutually exclusive and independent
12. A die is rolled and a coin is tossed. The probability of the die showing an even number and the coin coming up tails is
   A. \( \frac{1}{2} \)
   B. \( \frac{1}{4} \)
   C. \( \frac{1}{8} \)
   D. \( \frac{1}{12} \)

13. Four people roll a fair die to see who starts a game of scrabble. The probability that the first person throws a 2, the second person throws a 2, the third person throws a 4, and the fourth person throws a number other than 2 or 4 is
   A. \( \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{4}{6} \)
   B. \( \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{4}{6} \)
   C. \( \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{2}{6} \)
   D. \( \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{2}{6} \)

14. A triangular spinner has 3 equal sections coloured red, blue, and green. The probability, to the nearest hundredth, of the spinner not landing on red on four consecutive spins is _____.
**Probability Lesson #4:**

**Probability Problems Involving Independent Events**

### Warm-Up #1

**Review of Probability Formulas**

<table>
<thead>
<tr>
<th>If the events A, B, are <strong>mutually exclusive</strong>, then</th>
<th>If the events A, B, are <strong>NOT mutually exclusive</strong>, then</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ P(A \text{ or } B) = P(A) + P(B) ]</td>
<td>[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>If the events A, B, are <strong>independent</strong>, then</th>
<th>If the events A, B, are <strong>dependent</strong>, then</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ P(A \text{ and } B) = P(A) \times P(B) ]</td>
<td>[ P(A \text{ and } B) = P(A) \times P(B</td>
</tr>
</tbody>
</table>

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**Class Ex. #1**

Two hockey players, Wayne and Mario, each independently take a penalty shot. Wayne has a \( \frac{7}{10} \) chance of scoring, and Mario has a \( \frac{3}{5} \) chance of scoring. What is the probability that;

a) both score?  
b) both miss?  
c) only one of them scores?  
d) at least one of them scores?

---

**Class Ex. #2**

In the World Cup Soccer final, the score at the end of regular time was Brazil 2 Italy 2. After five penalty kicks for each team, the game was still tied. The game will now be decided by a penalty shootout where each team takes alternate shots on goal. Each team shoots once in each round. If both teams score, or both teams miss, they go on to another round. From past records the probability that Brazil will score on a penalty shot is 0.7 and the probability that Italy will score on a penalty shot is 0.6. What is the probability that;

a) Brazil wins in the first round  
b) Italy wins in the first round?  
c) Brazil wins in the second round?
Johann is planning a trip during the forthcoming three-day holiday weekend. He is considering cancelling the trip if the weather will be wet. The probability of a wet day is 0.3 and the weather on each day is independent of the weather on the other days.

\[ P(W_1 \text{ and } W_2 \text{ and } W_3) = \]

Complete the tree diagram to determine the probability that;

a) all 3 days will be wet

b) there will be only 1 wet day

c) that at least 2 days will be dry
Consider the following problem.

“Two friends Albert and Breanne are playing Scrabble. To decide who starts, they throw a die alternately until one of them throws a six. Albert throws first.”

Find the probability that Albert starts the game of Scrabble.

Let \( A_i \) be the event that Albert throws a six on his \( i^{th} \) attempt.

\( B_i \) be the event that Breanne throws a six on her \( i^{th} \) attempt.

Albert will start the game of Scrabble if any one of the following events occurs:

<table>
<thead>
<tr>
<th>Event</th>
<th>Probability Notation</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Albert throws a six on his first attempt</td>
<td>( P(A_1) )</td>
<td>( \frac{1}{6} )</td>
</tr>
<tr>
<td>or</td>
<td></td>
<td></td>
</tr>
<tr>
<td>neither Albert nor Breanne throws a six on their first attempt and Albert throws a six on his second attempt</td>
<td>( P(\overline{A}_1 \text{ and } \overline{B}_1 \text{ and } A_2) )</td>
<td>( \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \left( \frac{5}{6} \right)^2 \times \frac{1}{6} )</td>
</tr>
<tr>
<td>or</td>
<td></td>
<td></td>
</tr>
<tr>
<td>neither Albert nor Breanne throws a six on their first two attempts and Albert throws a six on his third attempt</td>
<td>( P(\overline{A}_1 \text{ and } \overline{B}_1 \text{ and } A_2) )</td>
<td>( \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \left( \frac{5}{6} \right)^2 \times \frac{1}{6} )</td>
</tr>
</tbody>
</table>

The probabilities in the extreme right column form an infinite geometric sequence.

a) Complete the next row of the table and use the results to determine the common ratio of the infinite geometric sequence.

b) Find the sum of the terms of this infinite geometric sequence to determine the probability that Albert starts the game.
Assignment

1. Alan and Steve each attempt a three-point shot in basketball. The probability that Alan is successful is 0.4, and the probability that Steve is successful is 0.35 and the events are independent. Calculate the probability that:

a) both players are successful
b) only one player is successful
c) at most one player misses.

2. Mike and Phil are playing in the final of a matchplay golf tournament. On the first hole, they each have a ten foot putt for birdie. The probability that Mike makes the putt is 0.6 and the probability that Phil makes the putt is 0.5, and these events are independent. Assuming that neither player will take more than two putts to complete the hole, calculate the probability that:

a) they both make the ten foot putt
b) Mike wins the hole
c) the hole is halved

3. In a Pee Wee hockey tournament a shootout consists of the Rebels and the Crusaders taking alternate shots on goal. Each team shoots once in each round. If both teams score, or both teams miss, they go on to another round. From past records the probability that the Rebels will score on a shot is 0.3 and the probability that the Crusaders will score on a shot is 0.2. What is the probability that:

a) Rebels win in the first round?
b) Crusaders win in the first round?
c) Rebels win in the second round?
4. Chess is a game played by two players. A game results in one of three mutually exclusive outcomes - either player A wins, or player B wins, or the game results in a draw. In a world championship chess match, the Russian champion Kozlov and the American champion Armstrong are involved in a sudden death playoff. In the playoff, Kozlov and Armstrong keep playing games until one of them wins a game. The first player to win a game wins the championship. From previous results, the probability that Kozlov wins a game is $\frac{1}{3}$ and the probability that Armstrong wins a game is $\frac{1}{4}$.

a) Explain why the probability that the first game is a draw is $\frac{5}{12}$.

b) Calculate the probability that the championship has not been decided after the second game.

c) Calculate the probability that Armstrong wins the championship within two games.

5. The probability that Andy solves a particular problem is $\frac{1}{3}$, that Barry solves that problem is $\frac{1}{2}$ and that Curtis solves the problem is $\frac{3}{5}$. Given these probabilities are independent, find the probability that the problem is solved by:

a) all three students  

b) none of the students  

c) only one of the students
6. The probability of being caught in a traffic jam in the morning rush hour is 0.6 on any particular week day. If the occurrence of traffic jams on different days are assumed to be independent of each other, find;

a) the probability that the journey is free from traffic jams for three consecutive week days.

b) the probability that there is a traffic jam on at least two out of three consecutive week days.

7. A student has difficulty in getting up for school, and so, to waken himself, he sets three alarm clocks to ring at the same time, as the noise from at least two alarms is necessary to waken him. Each alarm goes off independently. The probabilities that each alarm rings are 0.7, 0.8, and 0.9, respectively. Find the probabilities that:

a) all three alarms ring

b) no alarm rings

c) the student is awakened
8. Of the patients registered at a clinic, 40% attend Dr. Adams, 30% attend Dr. Barber, and the remainder attend Dr. Chang. For the next two patients, calculate the probabilities that:
   a) they are both for Dr. Chang   b) they are both for the same doctor.
   c) at least one is for Dr. Barber.
Assume that the choice of doctor is independent from patient to patient.

9. The school table tennis champion plays two opponents $A$ and $B$ in turn until one of them defeats him. The probability that the champion beats $A$ is 0.8, and the probability that the champion beats $B$ is 0.7. If the first game is against $A$, find the probability that it is $A$ who beats the champion.
10. Two hockey teams, the Spartans and the Burners, have a sudden death penalty shootout to decide who wins the game. The teams take penalty shots in turns. The first team to score wins. The probability that the Spartans score on any penalty shot is 0.3 and the probability that the Burners score on any penalty shot is 0.4. If the Spartans take the first penalty shot, determine the probability that they win the game.

**Answer Key**

1. a) 0.14  b) 0.47  c) 0.61
2. a) 0.3  b) 0.3  c) 0.5
3. a) 0.24  b) 0.14  c) 0.1488
4. a) There are three mutually exclusive outcomes with a total probability of 1.
   So the probability of a draw is \(1 - \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{5}{12}\)
   b) \(\frac{25}{144}\)
   c) \(\frac{17}{48}\)
5. a) \(\frac{1}{10}\)
   b) \(\frac{2}{15}\)
   c) \(\frac{2}{5}\)
6. a) 0.064  b) 0.648
7. a) 0.504  b) 0.006  c) 0.902
8. a) 0.09  b) 0.34  c) 0.51
9. \(\frac{5}{11}\)
10. \(\frac{15}{29}\)

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Probability Lesson #5: Probability Problems Involving Conditional Probability

Warm-Up #1  Review

Recall the following probability formulas:

- If the events \( A, B \), are **mutually exclusive**, then
  \[
  P(A \text{ or } B) = P(A) + P(B)
  \]
- If the events \( A, B \), are **NOT mutually exclusive**, then
  \[
  P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)
  \]
- If the events \( A, B \), are **independent**, then
  \[
  P(A \text{ and } B) = P(A) \times P(B)
  \]
- If the events \( A, B \), are **dependent**, then
  \[
  P(A \text{ and } B) = P(A) \times P(B \mid A)
  \]

The formula for dependent events can be written as \( P(A \mid B) = \frac{P(A \text{ and } B)}{P(A)} \)

Warm-Up #2  Review

One card is drawn at random from a deck of 52 cards. The following events are defined:
- \( A \): a diamond is drawn
- \( B \): a ten is drawn
- \( C \): a red card is drawn

Express the following probabilities as fractions in simplest form:

a) \( P(\overline{A}) = \) _____
b) \( P(A \text{ or } C) = \) _____
c) \( P(B \text{ and } C) = \) _____
d) \( P(A \text{ and } \overline{B}) = \) _____
e) \( P(A \mid C) = \) _____
f) \( P(C \mid A) = \) _____

Class Ex. #1

Two fair dice are rolled. Calculate the probability that 2 “ones” are rolled given that at least 1 “one” is rolled.
The table shows how the students in a large high school generally travel to school.

<table>
<thead>
<tr>
<th></th>
<th>Bus</th>
<th>Car</th>
<th>Other</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male, $M$</td>
<td>350</td>
<td>200</td>
<td>75</td>
<td></td>
</tr>
<tr>
<td>Female, $F$</td>
<td>300</td>
<td>175</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
    &\text{Male, } M \\
    &\quad 350 \\
    &\text{Female, } F \\
    &\quad 300 \\
    &\text{Total} \\
    &\quad \text{Total} \\
\end{align*} \]

a) Complete the totals in the chart.

b) How many students attended the high school?

c) If a student is selected at random, determine the probability that
   i) the student is female  ii) the student travels by bus  iii) the student is female and travels by bus

d) Determine the probability that:
   i) a female student travels by bus.  ii) a student who drives is male

e) Are the events “the student is female” and “the student travels by bus” independent events? Explain.

As part of an experiment into the learning process, a mouse is put into a maze and rewarded with food every time it turns right. If a mouse turns right, the probability it turns right the next time is increased by 20%. If a mouse turns left, the probability it turns left the next time is decreased by 20%. Assuming that there is an equal probability that the first turn will be to the left or right, calculate the probability that:

a) the first two turns are both right  b) the first two turns are both left

c) the first two turns are different  d) the first three turns are all left
Using a Probability Tree For Dependent Events

Cheryl is trying to show Jon how to solve problems based on the following information.

“Two machines $A_1$ and $A_2$ produce all the glass bottles made in a factory. Machine $A_1$ produces 60% of the output. The percentages of broken bottles produced by these machines are 5% and 8% respectively.”

Cheryl suggests the following strategy.

First: Introduce symbols to represent the information.

Second: Write the given probabilities in terms of the symbols.

Third: Set up a tree diagram.

a) Complete Jon’s work which is started below.

- $A_1$ - bottle is from machine $A_1$. $A_2$ - bottle is from machine $A_2$. $B$ - bottle is broken.

- $P(A_1) = \frac{\text{P}}{\text{A}_1} = \frac{\text{P}}{\text{A}_2} = \frac{\text{P}}{\text{B}} = 0.05 \quad \text{P}(\text{B}) = 0.08$

b) Jon sets up a probability tree diagram with the first branches leading to the machines and the second set of branches leading to the defective/non-defective items. Complete the diagram.

\[
P(B \mid A_1) = B \rightarrow P(A_1 \text{ and } B) = P(A_1) \cdot P(B \mid A_1)
\]

\[
P(A_1) = 0.6
\]

\[
P(B \mid A_1) = \frac{\text{P}}{\text{B}} \rightarrow P(A_1 \text{ and } \overline{B}) =
\]

\[
P(A_2) =
\]

\[
P(B) =
\]

C) Cheryl asked Jon some questions to see if he understands what he has drawn.

i) Which of the final outcomes in the diagram relate to the event “a bottle is broken”.

ii) If a bottle is chosen at random determine the probability that the bottle is broken.

iii) Write a formula for $P(B)$ in terms of conditional probabilities.
The formula on the previous page can be extended to include any number of machines \( A_1, A_2, A_3, \ldots \).

In general if a sample space is partitioned into mutually exclusive outcomes \( A_1, A_2, A_3, \ldots \) and if \( B \) is any other event then

\[
P(B) = P(A_1 \text{ and } B) + P(A_2 \text{ and } B) + P(A_3 \text{ and } B) + \ldots
\]

and hence

\[
P(B) = P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3) + \ldots
\]

This formula is called the formula on total probability and is NOT given on the formula sheet.

Consider the following problem.

“Bag A contains 5 yellow and 5 green marbles. Bag B contains 7 yellow and 3 green marbles. One of the bags is chosen by selecting one card at random from a deck of cards. If a heart is selected, then a marble is taken at random from Bag A. If a heart is not selected, then a marble is taken from Bag B.

What is the probability that the marble is yellow?”

a) Complete:

\[
P(A) = \quad P(B) = \quad P(Y \mid A) = \quad P(Y \mid B) =
\]

b) Rewrite the formula on total probability using the symbols in a) and solve the problem.

c) Use a tree diagram to solve the problem.
Two boxes each contain 6 doughnuts. The first box contains 3 chocolate doughnuts and 3 jelly doughnuts. The second box contains 2 chocolate doughnuts and 4 jelly doughnuts. One of the boxes is selected at random and a doughnut is removed. It is a jelly doughnut. A second doughnut is then removed from the same box.

Calculate the probability that it is a jelly doughnut.
Assignment

1. Two cards are drawn with replacement from a deck of cards. Calculate the probability that two queens are drawn given that at least one queen is drawn.

2. An octagonal die numbered 1-8 is rolled twice. What is the probability of rolling only one 8, given that at least one 8 is rolled?

3. The table shows the distribution of blood types for students in the first year at a local college.

\[
\begin{array}{|c|c|c|c|c|}
\hline
& O & A & B & AB \\
\hline
\text{Male, } M & 210 & 174 & 74 & 42 \\
\hline
\text{Female, } F & 315 & 261 & 111 & 63 \\
\hline
\text{Total} & & & & \\
\hline
\end{array}
\]

a) Complete the totals in the chart.

b) How many students are in first year at the college?

c) If a student is selected at random, determine the probability, to four decimal places, that the student;
   i) is male
   ii) has blood type A
   iii) is male and has blood type A

d) Are the events “the student is male” and “the student has blood type A” independent events? Explain.

e) Determine the probability, to four decimal places, that:
   i) a female student has blood type B
   ii) a female student does not have blood type O
   iii) a student with blood type A is male
   iv) a student with blood type AB is female.
4. Packet 1 contains 8 red balloons and 4 blue balloons. Packet 2 contains 5 red balloons, 3 green balloons and 4 blue balloons. A die is rolled. If the result is a one or a six then a balloon is chosen at random from packet 1. Otherwise, a balloon is chosen from packet 2. Calculate the probability that the chosen balloon is

a) blue

b) green

5. In a diagnostic test for a disease, sometimes a positive reaction is obtained, even when the disease is not present. In the past the test has given a positive reaction to 90% of people having the disease and to 5% of people who do not have the disease. If 10% of the population have the disease, calculate the probability that a person chosen at random would not react positively to the test.
6. Three machines A, B and C produce respectively 50%, 30% and 20% of the items produced daily by a manufacturing company. The percentages of defective items produced by the machines are respectively 5%, 2% and 1%. What is the probability that an item selected at random from the daily output is defective?

7. A golfer is practicing putting. The probability that he holes the first putt is 0.4. Each time he holes a putt the probability that he holes the next putt increases by 25%. Each time he misses a putt the probability that he misses the next putt increases by 20%. Calculate the probability that:
   a) he holes the first two putts
   b) he misses the first two putts
   c) he holes two of the first three putts
8. A box contains 10 cans of cola and 6 cans of lemonade. A second box contains 8 cans of cola and 8 cans of lemonade. One of the boxes is chosen at random and a can is selected at random from that box. The selected can is a can of lemonade. A second can is then selected from the same box.

Determine the probability that the second can is also lemonade.
9. A packet of candy contains 10 individually wrapped pieces of candy, each of which is either orange flavoured or lemon flavoured. One particular packet contains 8 orange and 2 lemon candies and a second packet contains 7 orange and 3 lemon candies. One of these packets is chosen at random and a candy is selected at random from that packet. It is lemon flavoured. A second piece of candy is then selected from the same packet. Calculate the probability that the second piece of candy is orange flavoured.

Answer Key

1. \(\frac{1}{25}\)
2. \(\frac{14}{15}\)
3. b) 1250 c) i) 0.4000 ii) 0.3480 iii) 0.1392 d) independent since \(P(M \text{ and } A) = P(M) \times P(A)\)
e) i) 0.1480 ii) 0.5800 iii) 0.4000 iv) 0.6000
4. a) \(\frac{1}{3}\) b) \(\frac{1}{6}\)
5. 0.865
6. 0.033
7. a) 0.2 b) 0.432 c) 0.2138
8. \(\frac{43}{105}\)
9. \(\frac{37}{45}\)

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Probability Lesson #6: Bayes’ Law

Warm-Up  Developing Bayes’ Law

Recall the following problem from Lesson #5:

“Cheryl is trying to show Jon how to solve problems based on the following information.

“Two machines $A_1$ and $A_2$ produce all the glass bottles made in a factory. Machine $A_1$ produces 60% of the output. The percentages of broken bottles produced by these machines are 5% and 8% respectively.”

Jon set up the following probability tree diagram to calculate the probability of a broken bottle.

Jon used the formula $P(B) = P(A_1 \text{ and } B) + P(A_2 \text{ and } B)$ (circled above)

$= P(A_1) P(B | A_1) + P(A_2) P(B | A_2) = 0.062$

Cheryl then posed the question.

“If a bottle selected at random is found to be defective, what is the probability it was produced by machine $A_1$?”

We are required to find $P(A_1 | B)$.

This can be done using the conditional probability formula

$$P(A_1 | B) = \frac{1^{st} \text{ branch}}{1^{st} \text{ branch} + 3^{rd} \text{ branch}} = \frac{P(A_1 \text{ and } B)}{P(B)} = \frac{P(A_1) P(B | A_1)}{P(A_1) P(B | A_1) + P(A_2) P(B | A_2)}$$

Complete the work to determine the answer.
Bayes’ Law

If a sample space is partitioned into mutually exclusive outcomes $A_1, A_2, A_3 \ldots$ and if $B$ is any other event then

$$P(A_1 \mid B) = \frac{P(A_1 \text{ and } B)}{P(B)} = \frac{P(A_1) \cdot P(B \mid A_1)}{P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \ldots}$$

Class Ex. #1

Urn A contains 4 black balls and 2 white balls. Urn B contains 4 black balls and 6 white balls. An urn is selected at random and a ball is drawn from that urn. If the ball is black, what is the probability it came from urn A?
A new test for detecting a disease has been developed. Medical trials show that 92% of the patients who have the disease react positively to the new test, while 5% of patients not suffering from the disease also react positively. If 4% of the population have the disease, determine the probability that:

a) a person reacts positively to the test.

b) a person showing a positive reaction suffers from the disease.
Assignment

1. A carton of tennis balls contains 4 yellow and 2 pink balls. A second carton contains 5 yellow and 1 pink ball. One of the cartons is selected at random and a ball removed. Given that the ball is pink, calculate the probability it came from the first carton.

2. Box A has 10 light bulbs of which 3 are defective. Box B has 8 light bulbs of which 2 are defective. A digit is selected at random from the digits 1 - 9. If the digit is even, a bulb is selected at random from box A. If the digit is odd, a bulb is selected at random from box B. Given that the bulb selected is defective, calculate the probability it came from box A.
3. A school cafeteria sells small baskets of fruit. At the start of a school day, the only fruit available was green or red apples and green or red grapes. A grade 12 student constructed the tree diagram shown to represent the probabilities associated with the baskets of fruit.

The first student at the cafeteria bought one of the small baskets of fruit. Determine the following probabilities associated with her choice.

a) $P(\text{apple})$

b) $P(\text{green apple})$

c) $P(\text{green} | \text{apple})$

d) $P(\text{apple} | \text{green})$

4. There are two bus routes between Luke’s home and downtown where he works. Luke uses route A two-thirds of the time, and route B one-third of the time. On route A, 10% of the buses are late, and on route B, 20% of the buses are late.

a) What is the probability that Luke is late for work on a particular day?

b) If Luke arrives late downtown, what is the probability he travelled by route A?
5. Box 1 contains 8 red dice and 4 blue dice. Box 2 contains 5 red dice and 7 blue dice. One card is randomly selected from a deck of playing cards. If the card is a diamond, a die is randomly selected from box 1. If the card is not a diamond, a die is randomly selected from box 2.

a) Find the probability that the die selected is red.

b) Given that the die selected is red, calculate the probability it came from box 1.

6. A high school student can walk, drive, or take the bus to school. The number of school days for each method of transport are in the ratio 3:2:1. If the student walks to school, the probability she is late is 0.4. If she drives to school the probability she is late is 0.2 and if she takes the bus the probability she is late is 0.1.

a) State the probability that on any particular day a student drives to school.

b) Construct a probability tree diagram and determine the probability that she is late on any particular day.
c) If she was late today, calculate the probability that she travelled by:
   i) walking
   ii) driving
   iii) bus

7. A large group of university students take three examinations in the course of a semester. It is found that if a student passes an examination, the probability of his passing the next one is four-fifths, and if he fails an examination, the probability of his failing the next one is also four-fifths. In the first examination, 60% of the students pass.

   a) What is the probability that a student passes all three examinations?
   b) If a student passes exactly two examinations, what is the probability that he failed the first one?
8. The following questions are extensions of questions in the assignment for Lesson #5. Use the tree diagrams from the assignments in Lesson #5 to answer the following.

a) on page 461, question #4

If the chosen balloon is red, calculate the probability it came from packet 2.

b) on page 461, question #5

If a person reacts positively to the test, what is the probability she actually has the disease?

c) on page 462, question #6

Given that the item is defective, calculate the probability it was produced by machine A.

---

**Answer Key**

1. $\frac{2}{3}$  
2. $\frac{24}{49}$  
3. a) $\frac{2}{3}$  
   b) $\frac{1}{6}$  
   c) $\frac{1}{4}$  
   d) $\frac{3}{7}$  
4. a) $\frac{2}{15}$  
   b) $\frac{1}{2}$  
5. a) $\frac{23}{48}$  
   b) $\frac{8}{23}$  
6. a) $\frac{1}{3}$  
   b) $\frac{17}{60}$  
   c) i) $\frac{12}{17}$  
   ii) $\frac{4}{17}$  
   iii) $\frac{1}{17}$  
7. a) $\frac{48}{125}$  
   b) $\frac{8}{23}$  
8. a) $\frac{5}{9}$  
   b) $\frac{2}{3}$  
   c) $\frac{25}{33}$

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**Probability Lesson #7:**
**Probability Involving Permutations and Combinations**

### Warm-Up

**Using Permutations or Combinations to Find the Probability of an Event**

Two cards are picked without replacement from a deck of 52 playing cards. Determine the probability that both are kings using

a) the multiplication law  

b) combinations

---

**Class Ex. #1**

The word **COUNTED** has been spelled using Scrabble tiles. Two tiles are randomly chosen one at a time and placed in the order in which they were chosen. Determine the probability that the tiles are:

a) CO  

b) both vowels

---

**Class Ex. #2**

The Athletic Council decides to form a sub-committee of seven council members to look at how funds raised should be spent on sports activities at the school. There are a total of 15 athletic council members, 9 males and 6 females. What is the probability that the sub-committee will consist of exactly 3 females?

---

**Class Ex. #3**

A bag of marbles contains 5 red, 3 green, and 6 blue marbles. If a child grabs three marbles from the bag, determine the probability that:

a) exactly 2 are blue  

b) at least one is blue

c) the first is red, the second is green and the third is blue  

d) one is red, one is green and one is blue
City Council consists of nine men and six women. Three representatives are chosen at random to form an environmental sub-committee.

a) What is the probability that Mayor Jim Milonovich and two women are chosen?

b) What is the probability that two women are chosen if Mayor Jim Milonovich must be on the committee?

In a card game you are dealt 5 cards from a pack of 52 shuffled cards. When you look at your 5 cards, what is the probability, expressed in combination notation, that you have:

a) four aces?  
b) four tens and an ace?

c) 10, J, Q, K and ace?  
d) at least one Jack?

In a class of 30 students, calculate the probability (to the nearest hundredth) that:

a) they all have different birthdays (assume no one is born on February 29)

b) at least 2 of them have the same birthday.

Complete Assignment Questions #1 - #14
Assignment

1. A marble is drawn at random from a box containing 10 red, 30 yellow, 20 blue, and 10 pink marbles. Find, as an exact fraction, the probability that the marble drawn is:
   a) yellow or red      b) not blue      c) green      d) red, pink or blue

2. A bag contains 6 blue marbles and 10 yellow marbles. Two marbles are drawn from the bag without replacement.
   a) Use the multiplication law to determine:
      i) \( P(\text{both are blue}) \)
   b) Use combinations or permutations to determine:
      i) \( P(\text{both are blue}) \)
      ii) \( P(\text{the first is blue and the second is yellow}) \)
      iii) \( P(\text{one is blue and one is yellow}) \)

3. Two marbles are drawn at random from a box containing 10 red, 20 yellow, 15 blue and 15 pink marbles. Find the probability, to the nearest thousandth, that
   a) both marbles are red      b) the first marble is red and the second marble is yellow
   c) neither marble is pink     d) both marbles are either blue or red
4. In a card game you are dealt 5 cards from a pack of 52 well shuffled cards. What is the probability, expressed in combination notation, that you have:
   a) four Queens? b) one face card? c) A, 2, 3, 4 and 5?
   
   d) at most one King? e) at least one diamond?

5. Three prizes are rewarded in a raffle during a halftime show at a school basketball game. Ben, Janelle, Jamie, and 17 other students each have one ticket.
   a) If the raffle has a first, second, and third prize, what is the probability, as an exact value, that Ben wins first prize, Janelle wins second prize, and Jamie wins third prize?
   
   b) If the raffle has three identical prizes, what is the probability that Ben, Janelle, and Jamie win the prizes?

6. Three toys are selected at random from a box of nine toys, three of which are defective. Find the probability (exact value) that of the toys selected:
   a) none are defective b) there are more defective than non-defective

7. A bank card personal identification number consists of any four digits. Repeat digits are allowed and the code can start with zero. What is the probability that a code begins and ends with 5?
8. In a small rural town there is only one football field to be shared by the five local teams. In the league competition, each team plays two games against every other team. What is the probability that the first game scheduled will have last year’s two best teams, Oilmen and Wheatmen in opposition?

9. A jar contains 32 balls, equal numbers of red, blue, green and yellow. Five balls are drawn from the jar, simultaneously and at random. Determine the probability (to the nearest thousandth) of each event
   a) There are three red balls and two yellow ones

   b) The balls are three of one colour and two of another colour

   c) All five balls have the same colour.

10. There are twelve boys and ten girls in an English 30 class. A yearbook committee of four is chosen at random.
   a) What is the probability, to the nearest hundredth, that Ryan, the school president, and three girls are on the committee?

   b) If Ryan must be on the committee, what is the probability, to the nearest hundredth, that three girls are chosen?
11. A six digit numeral is represented by the digits 1, 2, 3, 4, 5 and 6 in any order. If one of these numerals is chosen at random what is the probability that:

a) all the digits are in decreasing order?   b) the even digits are in decreasing order?

12. Chandra buys two of a total of thirty raffle tickets. There are two winning numbers. The probability that Chandra wins exactly one prize is

A. \( \frac{56}{435} \)  B. \( \frac{28}{435} \)  C. \( \frac{2}{435} \)  D. \( \frac{1}{435} \)

13. The letters of the word OKOTOKS are arranged. The probability, to the nearest hundredth, that an arrangement, chosen at random from all possible arrangements, begins and ends with a vowel is _____ .

14. The girls soccer team consists of 11 starters and 5 substitutes. The probability, to the nearest hundredth, that at least 2 of the 16 girls have the same birthday is _____ .

Answer Key

1. a) \( \frac{4}{7} \)   b) \( \frac{5}{7} \)   c) 0   d) \( \frac{4}{7} \)  2. i) \( \frac{1}{8} \)    ii) \( \frac{1}{4} \)    iii) \( \frac{1}{2} \)  3. a) 0.025   b) 0.056   c) 0.559   d) 0.169  4. a) \( \frac{4C_4 \cdot 48C_1}{52C_5} \)   b) \( \frac{12C_1 \cdot 40C_4}{52C_5} \)   c) \( \frac{(4C_1)^5}{52C_5} \)   d) \( \frac{4C_0 \cdot 48C_5}{52C_5} \)   e) \( \frac{4C_1 \cdot 48C_4}{52C_5} \)  5. a) \( \frac{1}{6840} \)   b) \( \frac{1}{1140} \)  6. a) \( \frac{5}{21} \)   b) \( \frac{19}{84} \)   c) \( \frac{1}{100} \)  7. \( \frac{1}{8} \)  8. \( \frac{1}{10} \)  9. a) 0.008   b) 0.093   c) 0.001  10. a) 0.02   b) 0.09  11. a) \( \frac{1}{720} \)   b) \( \frac{1}{6} \)  12. A  13. 0.14  14. 0.28

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Statistics and Probability Distributions Lesson #1: Descriptive Measures of Data in Statistics

**Statistics**

Statistics is the name given to the science of collecting and analyzing numerical data.

- **Collecting Data** - data can be collected in a variety of ways. Data can be collected from a population or from a sample of a population.

- **Population** - the complete collection of all items in a study.

- **Sample** - a portion picked out of the population by various methods (eg. random, stratified, destructive, etc.). Data collected from the sample is used to represent and make inferences about the population.

- **Analyzing Data** - once data has been collected, it can be analyzed in a variety of ways. One of these ways is to provide descriptive measures for the data collected.

**Descriptive Measures of Data**

Once data has been collected it can be described numerically using the following two measures - one of which gives the location of the data and the other describes the dispersion of the data.

**Measures of Central Tendency** - used to describe the average or central value of the data.

**Measures of Dispersion** - used to describe how spread out, or dispersed, the data is.

![Descriptive Measures of Data Diagram]

- **Measures of Central Tendency**
  - mean
  - median
  - mode

- **Measures of Dispersion**
  - range
  - standard deviation
  - others
Measures of Central Tendency

- **Mean** - the mean is calculated by adding all the data values and dividing by the number of data values.
  The mean of a *population* is denoted by the symbol \( \mu \).
  The mean of a *sample* is denoted by the symbol \( \bar{x} \).

  The mean of a population and the mean of a sample are both calculated using the same method.

- **Median** - the median is determined by sorting the data in numerical order to find the middle value.

- **Mode** - the mode is the data value which has the highest frequency of occurrence.

Class Ex. #1

Calculate the measures of central tendency indicated in each of the following.

a) Calculate the mean and mode for the following data

\[
11 \quad 10 \quad 19 \quad 8 \quad 15 \quad 15 \quad 14 \quad 6 \quad 12 \quad 17
\]

i) mean  

ii) mode

b) Calculate the median of each of the following sets of data.

*Situation 1: Odd # of Data values*  
\[
11 \quad 10 \quad 19 \quad 8 \quad 15 \quad 12 \quad 6 \quad 12 \quad 18
\]

*Situation 2: Even # of Data values*  
\[
11 \quad 10 \quad 19 \quad 8 \quad 15 \quad 13 \quad 6 \quad 13
\]

Class Ex. #2

If the mean of the data 10, 4, 7, 12, 9, 4, \( x \), 6 is seven, find the value of \( x \).
Measures of Dispersion

- **Range** - the range is the difference between the largest and smallest data values.

- **Standard Deviation** - the standard deviation is a measure that describes the variation, or spread, between the data values and the mean of the data.
  The standard deviation of a population is denoted by the symbol $\sigma$.
  The standard deviation of a sample is denoted by the symbol $s$.

The following formulas are used to calculate standard deviation.

Population Standard Deviation ($\sigma$) \hspace{2cm} Sample Standard Deviation ($s$)

$$\sigma = \sqrt{\frac{\sum(x - \mu)^2}{n}}$$

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

These formulas result in slightly different values depending on whether the data values are collected from a population or from a sample. In this course it will be assumed that all data values have been collected from populations. The symbol $\sigma$ will be used in every example to represent standard deviation. The symbol $s$ will not be used in this course.

---

### Class Ex. #3

The heights of the Cobras basketball team are given in the diagram.

a) Calculate the mean height, $\mu$.

b) State the range.

c) Calculate the standard deviation, $\sigma$, using the appropriate formula above. The chart has been drawn to aid the calculation.

<table>
<thead>
<tr>
<th>Height (cm)</th>
<th>Deviation from the mean ($x - \mu$)</th>
<th>Deviation squared ($x - \mu)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>170</td>
<td></td>
<td></td>
</tr>
<tr>
<td>182</td>
<td></td>
<td></td>
</tr>
<tr>
<td>193</td>
<td></td>
<td></td>
</tr>
<tr>
<td>193</td>
<td></td>
<td></td>
</tr>
<tr>
<td>212</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Using a Graphing Calculator to find Mean and Standard Deviation

A graphing calculator can be used to determine the mean and standard deviation of a set of data values. Use the following method for a TI-83 Plus:

1. Clear all lists by accessing the command “ClrAllLists”.
   
   Press \( \text{2nd} + 4 \text{ ENTER} \).

2. Access the lists by pressing \( \text{STAT} \text{ ENTER} \).

3. Enter the data values in L1.

4. Exit the lists using the QUIT command.

5. To calculate the mean, standard deviation, and other descriptive measures, access the CALC menu by pressing \( \text{STAT} \) then scrolling to CALC.

   Select the command “1-Var Stats” and then press \( \text{ENTER} \text{ ENTER} \).

Verify the results in Class Example #3 using a graphing calculator.

Refer to the information in Class Example #3. If each player grows by 5 cm in the next year describe the change to:

a) the mean

b) the standard deviation

Consider the histograms representing two sets of data values \( A \) and \( B \).

Explain which data set has;

a) the greater mean

b) the greater standard deviation
Mean and Standard Deviation from a Frequency Table

A graphing calculator can be used to determine the mean and standard deviation of data given in a frequency table.

The method is identical to the method on the previous page except for the following:

- In step 3 “Enter the data values in L₁ and the corresponding frequencies in L₂.”
- In step 5 after selecting “1-Var Stats” enter L₁, L₂, then press ENTER ENTER.

The frequency table shows the number of hits per game by a baseball player during the course of one month.

<table>
<thead>
<tr>
<th># of hits</th>
<th># of games (frequency)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Calculate the mean and standard deviation, to the nearest tenth, of the number of hits per game.

Complete Assignment Questions #1 - #12

Assignment

1. The maximum daily temperatures in Vancouver over a 7 day period are 7.2, 4.8, 4.8, 2.0, 4.1, 12.7, 16.8

   Calculate, to the nearest tenth, the mean, median, mode, range, and standard deviation.

2. The data shows the amount of principal left on home mortgage loans handled by a loan officer at a bank.

   10,000 39,500 51,140
   13,000 25,900 43,200
   75,400 30,900 123,800

   Calculate, to the nearest dollar, the mean and standard deviation of the data.
3. ITA, an independent testing agency, was contracted to check the lifetimes of two brands of light bulbs. The lifetimes, in hundreds of hours, of seven bulbs of each brand are given in the table.

<table>
<thead>
<tr>
<th></th>
<th>White Glo</th>
<th>Burn Bright</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9.5</td>
<td>10.3</td>
</tr>
<tr>
<td></td>
<td>8.1</td>
<td>6.0</td>
</tr>
<tr>
<td></td>
<td>9.0</td>
<td>8.7</td>
</tr>
<tr>
<td></td>
<td>9.3</td>
<td>8.6</td>
</tr>
<tr>
<td></td>
<td>8.4</td>
<td>9.5</td>
</tr>
<tr>
<td></td>
<td>8.6</td>
<td>10.8</td>
</tr>
<tr>
<td></td>
<td>8.7</td>
<td>7.7</td>
</tr>
</tbody>
</table>

Determine the mean and standard deviation of each data set and comment on the results.

4. Cricket, a game which is popular in Australasia, the Indian sub-continent, South Africa, and the West Indies, originated in England. The data shows the number of runs scored by two players, Wasim and Mushtaq, in various innings:

<table>
<thead>
<tr>
<th></th>
<th>Wasim</th>
<th>Mushtaq</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>56</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>106</td>
<td>77</td>
<td></td>
</tr>
<tr>
<td>68</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>82</td>
<td>49</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>38</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td></td>
</tr>
</tbody>
</table>

Totals 432 366

a) On average, who is the better player?

b) Which player is more consistent? Why?

5. The annual sales, in millions of dollars, for 28 toy department stores is listed.

a) Calculate the mean, $\mu$, and standard deviation, $\sigma$, of the data. Answer to the nearest tenth.

b) Calculate the percent of sales, to the nearest whole number, that are:

i) within one standard deviation of the mean, (i.e. in the interval $\mu \pm 1\sigma$).

ii) within two standard deviation of the mean, (i.e. in the interval $\mu \pm 2\sigma$).
6. In a card game, each card from two to ten is assigned a value according to the number on the card. Face cards are assigned a value of 10, and aces are assigned a value of 11. A player is dealt all the diamonds in the deck. Determine the mean and the standard deviation, to one decimal place, for the set of cards she has.

7. The “Postage Stamp”, the 8\textsuperscript{th} hole at Royal Troon Golf course, is one of the most difficult short par threes in championship golf. During a recent tournament, the scores at that hole in the final round are shown.

<table>
<thead>
<tr>
<th># of strokes</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>58</td>
</tr>
<tr>
<td>4</td>
<td>47</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

Calculate the mean and standard deviation, to the nearest tenth, of the number of strokes taken.

Use the following information to answer Questions #8 - #12.

Four local hockey teams, the Astros, the Bruins, the Celtics, and the Dynos have just returned from an international hockey tournament in Quebec. The histograms show the number of goals each team scored in the 14 games they played.
Multiple Choice

8. The histogram which represents data with the highest value for the mean is _______.
   A. Astros  
   B. Bruins  
   C. Celtics  
   D. Dynos

9. The histogram which represents data with the highest value for the standard deviation is _______.
   A. Astros  
   B. Bruins  
   C. Celtics  
   D. Dynos

Numerical Response

10. The mean number of goals scored by the Astros, to the nearest tenth, is _______.

11. The standard deviation of the number of goals scored by the Celtics, to the nearest tenth, is _______.

12. The standard deviation of the number of goals scored by the Bruins, to the nearest hundredth, is _______.

Answer Key

1. mean = 7.5, median = 4.8, mode = 4.8, range = 14.8, standard deviation = 4.9
2. mean = $45,871 standard deviation is $33,351
3. White Glo mean = 8.8, Bright Burn mean = 8.8, White Glo standard deviation = 0.46 Bright Burn standard deviation = 1.50
4. Wasim: mean = 43.2, standard deviation = 31.6 Mushtaq: mean = 36.6, standard deviation = 19.7
   a) Wasim is better on average because he has a higher mean score.
   b) Mushtaq is the more consistent player because his standard deviation is lower, i.e. there is less variation between Mushtaq’s scores.
5. a) \( \mu = 32.3, \sigma = 6.3 \) i) \( \frac{19}{28} = 68\% \) ii) \( \frac{27}{28} = 96\% \)
6. mean = 7.3, standard deviation = 2.9
7. mean = 3.6, standard deviation = 0.8
8. D 9. A
10. 3.1 11. 2.0 12. 0.86
**Statistics and Probability Distributions Lesson #2:**

**Probability Involving the Binomial Distribution**

---

**Probability Distribution**

A probability distribution is a complete listing of all possible outcomes of an experiment with their associated probabilities. A probability distribution can be displayed as a list, in a table, in the form of a graph such as a histogram, or by a formula. The sum of all the probabilities will be one.

---

**Warm-Up #1**  
**Listing a Probability Distribution**

A white die and a blue die are rolled. The outcomes are shown in the array.

<table>
<thead>
<tr>
<th>White Die</th>
<th>Blue Die</th>
<th>Sum (S)</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{1, 1}</td>
<td>2</td>
<td>1/36</td>
</tr>
<tr>
<td>1</td>
<td>{1, 2}</td>
<td>3</td>
<td>1/36</td>
</tr>
<tr>
<td>1</td>
<td>{1, 3}</td>
<td>4</td>
<td>1/36</td>
</tr>
<tr>
<td>1</td>
<td>{1, 4}</td>
<td>5</td>
<td>1/36</td>
</tr>
<tr>
<td>1</td>
<td>{1, 5}</td>
<td>6</td>
<td>1/36</td>
</tr>
<tr>
<td>1</td>
<td>{1, 6}</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>{2, 1}</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>{2, 2}</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>{2, 3}</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>{2, 4}</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>{2, 5}</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

12  | Total     |          |             |

- **a)** Use the table to complete the probability distribution for the variable $S$ which represents the sum obtained when two dice are rolled.

- **b)** State:
  - i) $P(S = 5)$
  - ii) $P(S > 9)$
  - iii) $P(S \leq 5)$

- **c)** Display the probability distribution in a histogram.
Binomial Experiment

A binomial experiment is an experiment with the following properties:

1. There are a fixed number, \( n \), of identical trials in the experiment.
2. Only 2 outcomes are possible at each trial. The outcome of each trial is classified as:
   • “success” (the occurrence of a specified event)
   • “failure” (the non-occurrence of a specified event)
3. The trials are independent.
4. In each trial, the probability of success, denoted by \( p \), and the probability of failure, denoted by \( q = 1 - p \), remains constant.
5. The variable is the number of successes in \( n \) trials.

Class Ex. #1
A die is rolled five times. We are interested in counting the number of times “two” appears in the five rolls.

a) How many trials does this experiment have? Are they identical?

b) If this is regarded as a binomial experiment, define the events “success” and “failure”.

c) State the probability of:
   i) success
   ii) failure

d) Do the probabilities of success and failure remain constant from trial to trial?

Warm-Up #2

A hockey player participates in a rapid shooting contest. With each shot, he either hits (H) or misses (M) a target. Using past records, the probability of hitting the target is 0.7.

a) The player shoots at the target five times. Complete the list of all possible arrangements which result in the player hitting the target exactly twice.
   H H M M M
   H M H M M

b) Write the number of arrangements of HHMMM in factorial notation.

c) Write the answer in b) in \( _nC_r \) notation.

d) Determine the probability of the event HHMMM.

e) Determine the probability of the event “the player hits the target exactly twice”.

f) Suppose the hockey player takes \( n \) shots at a different target where the probability of hitting the target is \( p \). Write down the formula for the probability of \( x \) hits in the \( n \) shots.
Binomial Distribution

A binomial distribution is the distribution of all possible outcomes and their related probabilities from a binomial experiment. The binomial distribution can be applied in probability situations using the binomial probability distribution formula.

\[ P(x) = \binom{n}{x} p^x q^{n-x} \]

*This formula is on the formula sheet*

where;
- \( n \) is the number of trials
- \( x \) is the number of success in \( n \) trials
- \( p \) is the probability of success in each trial
- \( q = 1 - p \) is the probability of failure in each trial

Class Ex. #2

Five dice are tossed. What is the probability of rolling:

a) exactly three twos?  

\[ n = \quad x = \quad p = \]

b) at least three twos?

Class Ex. #3

20% of patients taking a certain migraine drug suffer side effects. If the drug is given to eight patients, what is the probability that at most two suffer side effects, to the nearest thousandth?

Complete Assignment Questions #1 - #3
Using a Graphing Calculator to Solve Binomial Distribution Problems

A graphing calculator can be used to solve binomial probability distribution problems. To access the binomial probability distribution formula on a TI-83 graphing calculator use the following steps:

1. Access the distribution menu DISTR by pressing 2nd then VARS keys.

2. Select “binompdf” (binomial probability distribution function) by scrolling down to 0 and then pressing ENTER or by pressing the number 0 then ENTER.

3. The command “binompdf(” will appear on the screen. Use the following to enter in values of a binomial distribution:

   \[ \text{binomialpdf}( n, p, x ) \]

4. Press ENTER to solve.

Class Ex. #4

Use the features of a graphing calculator to verify the answer from Class Example #2a). “Five dice are tossed. What is the probability of rolling exactly three twos?”

Note

- To enter in more than one value for \( x \), use \( \{ \} \) brackets around the values for \( x \). For example, to enter \( x = 3 \) or 4 or 5 use binompdf( \( n, p, \{3, 4, 5\} \)).

- To determine the complete binomial probability distribution, i.e. for all values of \( x \), omit the \( x \) value(s) and use only binompdf( \( n, p \)).

- To find the sum of probabilities, in either of the above cases, use the sum (answer) command. Use 2nd STAT scroll across to MATH and down to “sum(”.

Class Ex. #5

Use the features of a graphing calculator to verify the answer from Class Example #2b). “Five dice are tossed. What is the probability of rolling at least three twos?”

Complete Assignment Questions #4 - #9
Assignment

1. Which of the following are binomial experiments?
   a) Five cards are selected with replacement from a deck of cards. We are interested in the number of clubs that appear.
   b) Five cards are selected without replacement from a deck of cards. We are interested in the number of clubs that appear.
   c) 100 voters were asked which of the three major political parties they supported. We are interested in the number of votes for each party.

2. Let the variable \( x \) represent the number of boys in a family of five children.
   a) Write the binomial probability distribution function in the form \( f(x) = n C_p^x q^{n-x} \).
   b) Complete the table to show the probability distribution and draw a histogram of the results.

<table>
<thead>
<tr>
<th># of boys (x)</th>
<th>Prob (P(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>

3. The probability that a college student will graduate is 0.4. Determine the probability that out of 5 students
   a) none will graduate  
   b) one will graduate  
   c) at least one will graduate

4. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random:
   a) none will be defective  
   b) one will be defective  
   c) at most two will be defective
5. Forty percent of the candies produced by SuperCandy Inc. are ‘soft centres’ and the remainder are ‘hard centres’. From a pile of assorted candies, the supervisor Susan chooses a handful of six candies. What is the probability that her selection contains four ‘hard centres’?

6. The probability that Yaro will score from a penalty shot is $\frac{1}{3}$. Find the probability, to the nearest hundredth, that he will score at least three goals in his next five penalty shots.

7. Individual baseball cards, chosen at random from a set of 20, are given away with bars of chocolate. Randy needs one more card to complete the set. He buys 5 bars of chocolate. What is the probability, to the nearest hundredth, that he completes the set?

8. Seven coins are tossed. What is the probability that:
   a) exactly 1 tail is tossed? 
   b) all heads are tossed?
   c) 2 heads and 5 tails or 3 heads and 4 tails are tossed?

9. Ninety percent of the residents of a large city, who are of voting age, are registered to vote. In a sample of ten residents, the probability, to the nearest hundredth, that at least eight are registered to vote is _____ .

**Answer Key**

1. a) Yes  
   b) No  
   c) No

2. a) $f(x) = 5C_x \left(\frac{1}{2}\right)^x$  
   b) $\left(0, \frac{1}{32}\right), \left(1, \frac{5}{32}\right), \left(2, \frac{5}{16}\right), \left(3, \frac{5}{32}\right), \left(4, \frac{5}{32}\right), \left(5, \frac{1}{32}\right)$

3. a) $\frac{243}{3125}$  
   b) $\frac{162}{625}$  
   c) $\frac{2882}{3125}$

4. a) $\frac{256}{625}$  
   b) $\frac{256}{625}$  
   c) $\frac{608}{625}$

5. $\frac{972}{3125}$

6. 0.21

7. 0.23

8. a) $\frac{7}{128}$  
   b) $\frac{1}{128}$  
   c) $\frac{7}{16}$

9. 0.93

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In the previous lesson we discussed the binomial probability distribution. In this lesson we will introduce and apply what is considered to be one of the most important probability distributions - the **normal distribution**.

It has been discovered that many observations of physical measurements such as length, volume, mass, time, etc. all have common characteristics in how their data is distributed. A histogram of the data results in a “bell shape”.

If we drew a frequency polygon on each of the above histograms we would obtain a curve similar to the one shown. This curve is referred to as the **Bell Curve** or the **Normal Curve**.
Warm-Up #2  Understanding the Normal Distribution Curve

The normal distribution curve can best be understood by a concrete example.

A manufacturer who makes 100W light bulbs is interested in determining the distribution of the lifetimes of the bulbs. After testing 44 bulbs, he calculated the mean life of the bulbs to be approximately 900 hours and the standard deviation to be approximately 50 hours. The lifetimes, in hours, of the 44 bulbs tested is shown.

767  849  845  830.5  835.1  840  849.9  851  851.4  854.8  860
899  898  894  872  874  875  880  881  882.3  885  899
901  903  903  905  908  910  915  919.8  920  922  925
949  932  903  922  950.4  950.9  962.7  975  980.3  997.4  1052

a) Complete the following table

<table>
<thead>
<tr>
<th>Interval (Hours)</th>
<th>&lt; 800</th>
<th>800 - 850</th>
<th>850 - 900</th>
<th>900 - 950</th>
<th>950 - 1000</th>
<th>&gt;1000</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Data Values</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>% of Data Values</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\frac{15}{44} = 34.09%$</td>
</tr>
</tbody>
</table>

b) Using $\mu = 900$ and $\sigma = 50$, write the following numbers and intervals in terms of $\mu$ and/or $\sigma$.

i) 950 ii) 850 iii) 800 iv) 975

v) 900 to 1000 vi) 850 to 1050

c) Complete the normal distribution curve based on the results for this Warm-up

- The curve above is based on one particular example and is a good approximation for the standard normal distribution curve shown on the next page.
The Standard Normal Distribution Curve

For population data with mean $\mu$ and standard deviation $\sigma$ the normal curve is denoted by $N(\mu, \sigma^2)$. There is a different probability distribution for each value of $\mu$ and $\sigma$.

In order to compare normal curves and to solve probability problems involving normal distributions, we convert the normal distribution curve given in a problem into the standard normal distribution curve.

The distribution represented by the standard normal distribution curve has a mean value of 0 and a standard deviation of 1, denoted by $N(0, 1)$.

The equation of the standard normal distribution curve is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

Verify the bell-shape of the curve using a calculator window $x: [-4, 4, 1] \quad y: [-0.1, 0.5, 1]$

The diagram shows the approximate area under the standard normal distribution curve sub-divided into regions each of width equal to one standard deviation. The percentage of the area under the curve in each region is indicated.

Complete the following:

1. _____% of the data is above the mean.
2. _____% of the data is within one standard deviation of the mean.
3. _____% of the data is within two standard deviations of the mean.
4. _____% of the data is within three standard deviations of the mean.
5. _____% of the data is between 1 and 2 standard deviations above the mean
6. _____% of the data is between 1 and 2 standard deviations below the mean
Basic Properties of the Standard Normal Distribution Curve

1. The total area under the curve is 1.
2. The normal curve extends infinitely to the left and right.
3. The normal curve is symmetrical about the mean, i.e. 50% of the area under the curve is to the left of the mean and 50% is to the right.
4. All the data is represented by the area under the curve.
5. The mean, median and mode are the same value.

In post secondary studies you may come across data that is skewed, in which case some of the above points no longer apply.

Class Ex. #1

A nurse records the number of hours an infant sleeps during a day. He then records the data on a normal distribution curve shown below. The values shown on the horizontal axis differ by one standard deviation.

a) What is the mean of the data?  
b) What is the standard deviation?

c) What are the values for A, B, C, and D?

d) What percentage of a day, to the nearest hundredth, does the infant sleep:
   i) between 12 and 14 h?    ii) between 8 and 16 h?    iii) less than 6 h?

  e) Why is it not possible at this time to determine the percentage of a day that the infant sleeps for less than 13 hours?

To answer part e) we need the concept of z-scores.
A **z-score** for a data value describes the number of standard deviations above or below the mean.

### Displaying z-scores on the Normal Curve

Consider Class Example #1 with a mean value of 12 and a standard deviation of 2.

- The data value 16 is ____ standard deviations above the mean and has a **z-score** of ____ .
- The data value 6 is ____ standard deviations below the mean and has a **z-score** of ____ .
- The data value 12 is ____ standard deviations away from the mean and has a **z-score** of ____ .
- The data value 13 is ____ standard deviations _______ the mean and has a **z-score** of ____ .

![Data Values (x)](data_values.png)

\[ x \text{ in terms of } \mu, \sigma \]

\[ z \text{-scores} \]

### z-score Formula

**z-scores** can be calculated using the formula

\[
 z = \frac{x - \mu}{\sigma}
\]

where,
- \( z \) is the **z-score**
- \( x \) is the particular data value
- \( \mu \) is the mean
- \( \sigma \) is the standard deviation

*This formula is on the formula sheet*
The heights, in centimetres, of five starting members of the Wolfhounds basketball team are 170, 182, 193, 195, and 212. If the mean height is 190 cm and the standard deviation is 15 cm, calculate the \( z \)-score, to the nearest hundredth, for each data value. The first one is shown.

\[
z = \frac{x - \mu}{\sigma}
\]

\[
z_{170} = \frac{170 - 190}{15} = -1.33
\]

Tony’s midterm marks are shown below, together with the class mean and standard deviation for each subject. By calculating \( z \)-scores, determine in which subject Tony performed best relative to the rest of the class.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Tony’s Mark</th>
<th>Mean Mark</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics</td>
<td>74</td>
<td>68</td>
<td>12</td>
</tr>
<tr>
<td>Chemistry</td>
<td>79</td>
<td>73</td>
<td>14</td>
</tr>
<tr>
<td>Physics</td>
<td>68</td>
<td>66</td>
<td>11</td>
</tr>
</tbody>
</table>

The marks on a math exam at a university were found to have a mean of 52 with a standard deviation of 12. A professor who thought the exam was too difficult, decided to adjust the original marks by raising the mean to 65, while reducing the standard deviation to 10 and leaving the \( z \)-scores unchanged. What would the new mark be for a student who received an original mark of 34?

Complete Assignment Questions #1 - #6
Recall the standard normal distribution curve shown. The numbers on the horizontal axis represent $z$-scores.

The $z$-score tables give the area to the left of a particular $z$ value as shown below. This area to the left of $z$ is denoted by $A(z)$ or $F_z(z)$.

### Properties of $z$-scores

1. A $z$-score for a data value describes the number of standard deviations above or below the mean.
2. A negative $z$-score indicates that the data value is below the mean and is shown to the left of the mean on the standard normal distribution curve.
3. A positive $z$-score indicates that the data value is above the mean and is shown to the right of the mean on the standard normal distribution curve.
4. The $z$-score table gives the
   - area under the curve, to the left of the $z$-score, or
   - percentage of data to the left of the $z$-score, or
   - probability that a randomly chosen data value is to the left of the $z$-score.
5. The mean, median, and mode have a $z$-score of zero.

### Class Ex. #5

**a)** Use the $z$-score tables to calculate
- i) $A(-3)$
- ii) $A(-2)$
- iii) $A(1)$

**b)** Verify the answers using the standard normal distribution curve shown above.
Refer to Class Example #1e). Determine \( A(0.5) \) and use this to calculate the percentage of days that the infant slept for less than 13 hours.

1. Find the area under the standard normal curve for each \( z \)-score interval. Give the area as a decimal and as a percent. Label the diagram.
   a) \( z < 1.78 \)
   b) \( z > 2.61 \)

2. Determine the following probabilities
   a) \( P(-1.83 < z < 2.65) \)
   b) \( P(1.83 < z < 2.65) \)

For each of the following normally distributed curves find the \( z \)-score intervals which represent the area or percent given.

a) \( 17.62\% \)
   \( z \)-score

b) \( 14.92\% \)
   \( z \)-score

c) 0.2580
   \( z \)-score 0

d) 0.2145 0.1293
   \( z \)-score 0

Complete Assignment Questions #7 - #12
Assignment

1. The goals scored by a minor league hockey player over 7 seasons are shown below. Assuming the data is normally distributed, calculate the $z$-score to the nearest hundredth for the highest and lowest number of goals. (Round off the mean and standard deviation to the nearest tenth.)

   $35, 56, 47, 58, 63, 68, 75$

2. Joel’s unit test marks are shown below, together with the class mean and standard deviation for each unit test. By calculating $z$-scores, determine in which unit test Joel performed

   a) best  
   b) worst  
relative to the rest of the class.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Joel’s Mark</th>
<th>Mean Mark</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transformations</td>
<td>82</td>
<td>73</td>
<td>12</td>
</tr>
<tr>
<td>Logarithms</td>
<td>81</td>
<td>71</td>
<td>13</td>
</tr>
<tr>
<td>Trigonometry</td>
<td>74</td>
<td>64</td>
<td>14</td>
</tr>
<tr>
<td>Permutations</td>
<td>79</td>
<td>68</td>
<td>13</td>
</tr>
</tbody>
</table>

3. On an Science proficiency exam at a Canadian University, the mean score was 64 and the standard deviation was 9. If Geoff’s $z$-score was 1.3, then what was his actual exam mark?

4. A university test was given where the scores are normally distributed. A student had a score of 63% which was 2.13 standard deviations above the mean. If the mean of the exam was 57%, then what is the standard deviation to the nearest hundredth?
5. The marks on a math exam at a university were found to have a mean of 55 with a standard deviation of 14. A professor who thought the exam was too difficult, decided to adjust the original marks by raising the mean to 63, while reducing the standard deviation to 12 and leaving the \( z \)-scores unchanged. What would the new mark be for a student who received an original mark of 37?

6. The mean on a statistics multiple choice exam is \( 10k \) with a standard deviation of \( 2k - 4 \). On this exam, the student’s mark is represented by a \( z \)-score of 2, and his actual mark is represented by \( 16k - 32 \). What is the student’s actual score?

7. Find the area under the standard normal curve for each \( z \)-score interval. Give the area as a decimal and as a percent. Label the diagram.
   a) \( z < -1.6 \)  
   b) \( z > -1.6 \)  
   c) \( z > 1.6 \)  
   d) \( -0.31 < z < 1.56 \)  
   e) \( 0.31 < z < 3.32 \)
8. Find the probability of each
   
a) \( P( -2.34 < z < -1.3 ) \)  
b) \( P(z > -1.32) \)  
c) \( P(z < -2.42) \)

9. For each of the following normally distributed curves find the \( z \)-score interval which represents the area given.

   a)
   ![Diagram](image1)

   b)
   ![Diagram](image2)

   c) \( z \)-score: 0.2995

   d)
   ![Diagram](image3)

   e) \( z \)-score: 0.1293

   f)
   ![Diagram](image4)

   g) \( z \)-score: 0.4236

   h)
   ![Diagram](image5)
10. Find the value of $a$.
   
   a) $P(z < a) = 0.1379$
   
   b) $P(z > a) = 0.8508$
   
   c) $P(0 < z < a) = 0.3907$
   
   d) $P(a < z < 0) = 0.4306$

11. The shaded area in the diagram represents 23.97% of the area under the curve.
    The value of $z_1$ is
    
    A. $-0.92$
    B. $-0.71$
    C. $-0.64$
    D. $-0.52$

12. The area, to the nearest tenth of a percent, under the standard normal distribution curve which lies within 1.5 standard deviations of the mean is _____.
A light bulb manufacturer produces 35,000 light bulbs. From past experience the mean life of the light bulbs is 900 hours and the standard deviation is 50 hours. Determine:

a) the percentage of light bulbs that will last between 825 and 875 hours

b) the number of light bulbs that will last between 825 and 875 hours

c) the probability that a bulb selected at random will last less than 920 hours.

A study showed that the mean duration of a certain strain of flu virus was 12 days with a standard deviation of 3 days. If you caught this type of flu, what is the probability, to the nearest hundredth, that it would last:

a) longer than 17 days? 

b) between 13 and 15 days?
From extensive testing, an appliance distribution company knows that the average life of “Toasty” toasters is 4.2 years and the standard deviation is 0.65 years. The company does not want to replace under warranty more than 8% of the toasters that are sold. What guarantee, to the nearest necessary year, should the company offer?

It was found that 37.7% of the shrimp harvested at Shrimp Harvest Farms had a mass of less than 135 grams. If the mean mass of the shrimp harvested was 146 grams, determine the standard deviation to the nearest tenth.

The marks of a large number of students have been represented on a standard normal distribution curve. The values given represent the number of students in each area. Determine the value of \( z_1 \) to the nearest hundredth.
Assignment

1. The results of a provincial grade nine achievement test were normally distributed with a mean of 68 and a standard deviation of 12. If 8500 students wrote the test, determine:
   a) the percentage of students, to the nearest tenth of a percent, who scored a mark of 50 or above
   b) the number of students who scored a mark of 50 or above.
   c) the probability that a student selected at random had a mark
      i) less than 30
      ii) between 50 and 60

2. The “Long Life” battery company is planning to add another 10 500 batteries to their yearly production of batteries. If the mean life of “Long Life” batteries is estimated to be 50 h with a standard deviation of 6 h, then how many of the new batteries to be produced can be expected to last less than 31 hours?

3. The heights of 800 officers from a police force are normally distributed with a mean of 175 cm and a standard deviation of 8 cm.
   a) How many of the officers are within one standard deviation of the mean height?
   b) How many of the officers are between 167 and 173 cm?
   c) What percentage of the officers are between 165 cm and 180 cm?
4. Data collected of cars passing on a road revealed that the average speed was 90 km/h with a standard deviation of 5 km/h and data which is normally distributed. A policeman is assigned to set photo radar on a road in which the posted speed limit is 80 km/h. If 600 cars pass the photo radar and the policeman sets the camera so that only those exceeding the speed limit by 10% are photographed, then how many drivers can the police expect to ticket?

5. The probability of the shaded area is 0.2397. What is the value of $z_1$?

6. The results of a provincial achievement test were normally distributed and shown below. The data below the curve represents all of the students who wrote the test. The values 452 and 2500 represent the number of students in the shaded regions.

   a) How many students wrote the test?  
   b) What is the value of $z_1$?

7. The data below the curve is normally distributed and represents the number of diamonds which are exported monthly. What is the value of $z_1$?
8. What is the probability, to the nearest tenth of a percent, that a data value lies between the intervals $\mu - 0.85\sigma$ and $\mu + 2.31\sigma$?

9. After reviewing previous loan records, the credit manager of a bank determines that the data follows a normal distribution. The debts have a mean of $20,000 and the probability that the loss could be greater than $25,000 or less than $15,000 is 0.418. Determine the standard deviation of the data to the nearest hundred dollars.

10. The “Live Long” line of nine-volt batteries are normally distributed. If 9.68% of the batteries last more than 46 hours and 24.2% last less than 42 hours, what are the mean and standard deviation of the lifetimes of this brand of batteries? (Hint: try obtaining two equations and use a system of equations to solve).
11. What symmetrical z-score interval contains 69.2% of the data under a standard normal distribution curve?

12. A star offensive hockey player averaged 611 shots per season with a standard deviation of 57 shots. If the player played for 16 seasons, then the number of seasons the player had at least 675 shots is _____.

13. The weights of a large shipment of coconuts are normally distributed with a standard deviation of 0.71 kg. The probability that a coconut weighs less than 2.1 kg is 10.2%. The mean of the shipment, to the nearest hundredth, is _____.
Statistics and Probability Distributions Lesson #5: Using a Graphing Calculator to Solve Normal Distribution Problems

A graphing calculator can be used to solve SOME normal probability distribution problems.

The command “normalcdf” can be used to calculate normal distribution probability between two data values or to the left or right of a data value for a specified mean, \( \mu \), and standard deviation, \( \sigma \).

To solve problems using “normalcdf” on a TI-83 Plus graphing calculator use the following steps:

1. Access the distribution menu DISTR by pressing 2nd then VARS keys.

2. Select “normalcdf” (normal cumulative distribution function) by scrolling down to 2 and then pressing ENTER or by pressing the number 2 then ENTER.

3. The command “normalcdf(” will appear on the screen. Use the following to determine the area between two data values:

\[
\text{normalcdf( lower bound, upper bound, mean, standard deviation )}
\]

4. Press ENTER to determine the area.

- To calculate the area to the left of a data value, replace the lower bound by \(-1 \times 10^{99}\).  
- To calculate the area to the right of a data value, replace the upper bound by \(1 \times 10^{99}\).  
- The answers obtained will not be exactly the same as those obtained from tables due to the increased accuracy provided by the calculator.

Class Ex. #1

Use the features of a graphing calculator to verify the answer from Lesson #4 Class Example #2.

“A study showed that the mean duration of a certain strain of flu virus was 12 days with a standard deviation of 3 days. If you caught this type of flu, what is the probability, to the nearest hundredth, that it would last:

a) longer than 17 days?  
b) between 13 and 15 days?”
invNorm( \\

Given the area to the left of a data value, the command “invNorm(” can be used to calculate the data value. The mean and standard deviation must be given.

To access the normal probability distribution formula on a TI-83 Plus graphing calculator use the following steps:

1. Access the distribution menu DISTR by pressing 2nd then VARS keys.

2. Select “invNorm(” (inverse normal cumulative distribution function) by scrolling down to 3 and then pressing ENTER or by pressing the number 3 then ENTER.

3. The command “invNorm(“ will appear on the screen. Use the following to determine the data value:

\[
\text{invNorm( area, mean, standard deviation )}
\]

4. Press ENTER to determine the data value.

Use the features of a graphing calculator to verify the answer from Lesson #4, Class Example #3.

“From extensive testing, an appliance distribution company knows that the average life of “Toasty” toasters is 4.2 years and the standard deviation is 0.65 years. The company does not want to replace under warranty more than 8% of the toasters that are sold. What guarantee, to the nearest necessary year, should the company offer?”

Complete Assignment Questions #1 - #8

Assignment

1. The marks on an examination are normally distributed with a mean of 62 and a standard deviation of 10. Determine the probability, to four decimal places, that a student chosen at random has a mark

   a) less than 50   b) between 50 and 60   c) greater than 80
2. The heights of soldiers in a Canadian Army regiment are normally distributed with a mean of 173 cm and a standard deviation of 12 cm. The tallest 33 per cent of the soldiers in the regiment are eligible to drive a specially designed sub terrain vehicle. What is the minimum height, to the nearest hundredth of a cm, required to drive this specially designed vehicle?

3. A nut bolt with a circular opening is rejected if its diameter is greater than 2.01 cm or less than 1.97 cm. What is the expected number of rejected nut bolts if the diameters in a batch of 15 000 bolts are normally distributed with a mean of 2 cm and a standard deviation of 0.01 cm?

4. A manufacturer does a study on the photo radars they produce and find that they have a mean life of 16.3 years with a standard deviation of 4.2 years. If the data is normally distributed, then:
   a) What guarantee, to the nearest necessary year, should the manufacturer give so that fewer than 9% of the photo radar units would be returned?

   b) What is the probability, to four decimal places, that a police purchasing department will select, at random, a photo radar unit that needs to be returned within 8 years?

5. To protect their colourful paints, costume jewellery is sprayed with a translucent protective coating. A company which supplies the protective coating has determined the mean life of protection is 60 months with a standard deviation of the normally distributed data being 5 months.
   a) What guarantee, to the nearest necessary year, should the company which supplies the protective coating give so that less than 5% of the items sold will be returned under the guarantee?

   b) What is the probability, to the fourth decimal place, that costume jewellery chosen at random, will not have to be returned within 3.5 years?
6. At the Department of Statistics located at Range City, employees are encouraged to car pool for environmental reasons. Miss Median and Mr. Data, who work at the Department of Statistics, are two such employees who car pool. The mean time and standard deviation required for the trip from their apartment building to the office is 25 minutes and 5 minutes respectively. If the data is normally distributed, at what time to the nearest minute should they leave the apartment building to give themselves a 98.5% chance that they will be at work by 8 a.m.?

7. To determine what guarantee to give on the heads of video machines, a manufacturer collects the following data which is normally distributed. The mean life of the heads is 8.3 years and the standard deviation is 1.1 years.

a) If the manufacturer feels it is reasonable to replace 12 video machines for every 625 sold under warranty, then what guarantee, to the nearest year, should he offer on the heads of the video machines?

b) If the manufacturer plans to produce 10 000 video machines this year, then what is the probability, to four decimal places, that a machine will not be returned to replace the heads within 5 years?

8. After keeping excellent records from the past, Air Blow Manufacturing Company knows that its air compressors have an average working period of fifteen years with a standard deviation of 2.4 years. The manufacturer guarantees these compressors for 7 years. Assuming a normal distribution, the percentage, to the nearest hundredth of a percent, of compressors that the manufacturer will have to repair under warranty is _____ .

**Answer Key** These answers have been calculated using normalcdf or invNorm. Results obtained from z-score tables may vary slightly.

1. a) 0.1151  b) 0.3057  c) 0.0359  2. 178.28 cm  3. 2400 bolts
2. a) 10 years  b) 0.0241  5. a) 4 years  b) 0.9998  6. 7:24 a.m.
3. a) 6 years  b) 0.9987  8. 0.04
Statistics and Probability Distributions Lesson #6:  
The Normal Approximation to the Binomial Distribution

Binomial Experiment

Recall the conditions for an experiment to be considered a binomial experiment.

A binomial experiment is an experiment with the following properties:
1. There are a fixed number, \( n \), of identical trials in the experiment.
2. Only 2 outcomes are possible at each trial. The outcome of each trial is classified as:
   • “success” (the occurrence of a specified event)
   • or “failure” (the non-occurrence of a specified event)
3. The trials are independent.
4. In each trial, the probability of success, (denoted by \( p \)), and the probability of failure, (denoted by \( q = 1 - p \)) remains constant.
5. The variable is the number of successes in \( n \) trials.

Warm-Up #1a

Consider the following problem:

“ A teacher gives a short quiz consisting of 7 multiple choice questions. Each question has three answers, one of which is correct.
What is the probability that a student passes the quiz (a mark \( \geq 50\% \)) by guessing each of the answers?”

This question satisfies the conditions of a binomial experiment where the number of successes is the number of correct answers.
We require to find the probability of 4 or more successes where \( P(x) = nC_x p^x q^{n-x} \).
• Complete the solution.

Warm-Up #1b

Consider the following problem:

“ A teacher gives a short quiz consisting of 45 multiple choice questions. Each question has three answers, one of which is correct.
What is the probability that a student passes the quiz (a mark \( \geq 50\% \)) by guessing each of the answers?”

In this case we require to find the probability of 23 or more successes.

Using the binomial distribution to solve this problem would be tedious and an alternative method which provides an approximate answer is used. This method will be developed in the next two pages and the answer to this problem will be given.
Mean and Standard Deviation for the Binomial Distribution

We would expect students who guess all the answers to get, on average, one-third of the answers correct.

The expected number of correct answers is \( 45 \times \frac{1}{3} = 15 \).

This is an example of the formula \( \mu = np \) for the mean of the binomial probability distribution.

The standard deviation for the binomial probability distribution is given by the formula \( \sigma = \sqrt{npq} \).

Mean and Standard Deviation of the Binomial Probability Distribution

\[
\mu = np \\
\sigma = \sqrt{npq}
\]

These formulas are on the formula sheet

A shipment of 600 parts is received from a supplier. From past experience, it is known that the probability that any particular part is defective is 0.04. Calculate the mean and standard deviation, to the nearest tenth, of the number of defective parts in the shipment.

The Normal Approximation to the Binomial Distribution

Warm-Up #2

Consider the following problem:

“A teacher gives a short quiz consisting of 12 true/false questions. What is the probability, to four decimal places, that a student gets 10 or more answers correct by guessing each of the answers?”

a) Solve the problem using the binomial probability distribution.

b) Determine the mean and standard deviation of the number of correct answers.

c) Assume that the distribution of correct answers is normally distributed with the same mean and standard deviation as in b). Solve the problem using the normal probability distribution and compare the answer with a).
Using the normal approximation to the binomial distribution has in this case produced a result which is correct only to the first decimal place compared with the actual result obtained from the binomial distribution.
The accuracy could be improved by using a **continuity correction**.

### Continuity Correction

In problems involving the normal approximation to the binomial distribution, a **continuity correction** will be used to improve the accuracy of the approximation.

The binomial distribution is defined for discrete values of the variable (whole numbers). The normal distribution is defined for all values of the variable in a given domain and is therefore a continuous distribution. In using the normal distribution to approximate the binomial distribution we are approximating a discrete variable by a continuous one.

The probability that a binomial variable has a value 10 is approximated by the probability that the normally distributed variable has a value in the interval from 9.5 to 10.5.

In Warm-Up #2
- the **binomial** probability of 10 or more successes
  - is approximated by
  - the probability that the **normally distributed** variable is greater than or equal to 9.5

- Complete the solution and compare the answer with the solution obtained using the binomial distribution.

### Large Samples

In general, the normal approximation to the binomial distribution will be more exact the larger \( n \) becomes and the closer \( p \) is to 0.5.

In this course we will use the normal approximation only if the following two conditions are met:

\[
np > 5 \quad \text{and} \quad nq > 5
\]

These conditions are **NOT** on the formula sheet

If these conditions are met we are said to be using the normal approximation to the binomial distribution for **large samples**.

Note that some texts use the condition \( npq > 10 \).
Solve the problem in Warm-Up #1b) using the normal approximation to the binomial distribution.

“A teacher gives a short quiz consisting of 45 multiple choice questions. Each question has three answers, one of which is correct. What is the probability, to four decimal places, that a student passes the quiz (a mark ≥ 50%) by guessing each of the answers?”

Apples in a store contain 8% that are damaged. The apples are packed in boxes of 150 and priced for sale. If a customer buys one of these boxes, determine the probability, to the nearest hundredth, that the box contains less than 15 damaged apples.

Complete Assignment Questions #1 - #8
Assignment
1. In which of the following binomial situations is the normal approximation considered suitable. For those which are suitable, determine the mean and standard deviation of the approximating normal distribution and solve the problem.
   Give answers to the nearest hundredth where necessary.
   a) A die is rolled 25 times. Determine the probability of more than ten sixes.
   b) One hundred coins are tossed. Determine the probability of less than 45 heads.
   c) 90% of students are in favour of free school parking. Determine the probability that, in a class of 35 students, more than 30 are in favour of free school parking.

2. 70% of graduates of high schools in a large city take post secondary courses.
   a) Determine the probability that, in a random selection of 80 high school graduates, at least 60 are taking post secondary courses. Answer to the nearest hundredth.
   b) Determine the probability that, in a random selection of 150 high school graduates, 100 or less of them are taking post secondary courses. Answer to the nearest hundredth.
3. A die is rolled 720 times. Determine the probability, to the nearest hundredth, that a six will occur;
   a) between 100 and 125 times inclusive    b) at most 100 times

4. Of the voters in a certain town 60% are in favour of building a ring road to by-pass the town. A random sample of 104 voters is interviewed. Calculate the probability, to the nearest hundredth, that;
   a) at least two-thirds of the sample are in favour of the ring road

   b) a majority of the sample are not in favour of the ring road
5. In a certain culture the probability of a male birth is 0.52. Determine the probability, to the nearest hundredth, that there will be fewer males than females in 125 births.

6. 70% of students in an elementary school are bussed. In a sample of 50 students determine the probability, to the nearest hundredth, that the number of students who are bussed is

   a) between 20 and 30 inclusive  
   b) at least 40
7. 10% of the items produced in a factory are defective. In a sample of 400 items, the mean and standard deviation for the number of non-defective items are respectively

A. 40, 6  
B. 40, 36  
C. 360, 6  
D. 360, 36

8. A computer generates a series of random digits. In a sample of 350 random digits, the probability, to the nearest tenth, that there are at most 30 zeros is _____.

**Answer Key**

1. a) no since \( np = \frac{25}{6} \) which is less than 5  
   b) yes, mean = 50, standard deviation = 5, probability = 0.14  
   c) no since \( nq = 3.5 \) which is less than 5.

2. a) 0.20  
   b) 0.21

3. a) 0.69  
   b) 0.03

4. a) 0.08  
   b) 0.01

5. 0.33

6. a) 0.08  
   b) 0.08

7. C  
8. 0.2
Conic Sections Lesson #1:
Introducing Conic Sections

The Double-Napped Cone

A cone is a solid which can be generated by rotating a right angled triangle about one of its legs.

A double-napped cone is formed when the vertices of two cones are placed together. For example, we can make a double-napped cone by:

- taking two water cups shaped as cones and placing their tips against each other, or,
- placing a lecture pointer stick between your thumb and finger, holding it vertically in front of you, and then rotating it so that the top of the pointer and the bottom of the pointer form circles.

In general, a double-napped cone is produced by rotating an oblique line (called the generator) about an axis.

If the line is parallel to the generator, a cylinder is formed. This case will be discussed later.

A double-napped cone consists of the following parts: generator, upper nappe, lower nappe, vertex, axis of symmetry (called the central axis), generator angle, and vertex angle.

Label the following on the diagram.

a) the generator
b) upper nappe
c) lower nappe
d) vertex
e) axis of symmetry
f) generator angle
g) vertex angle
**Conic Sections**

Conic sections are two-dimensional figures which can be formed by a plane slicing a double-napped cone (or a cylinder).

Much of the work in this area of mathematics was discovered by the Greek mathematician Apollonius in about 200 B.C. He discovered that the intersection of a plane and a double-napped cone could result in one of four different conic sections, called the **primary conic sections**, according to the angle of intersection.

---

**The Primary Conics Generated from a Double-Napped Cone**

The angle of intersection between a cutting plane and a cone is defined as the angle between the central axis and the cutting plane.

If we take a plane and cut a double-napped cone at different angles to the axis and not through the vertex, we generate the **primary conics** illustrated below.

**Circle**  If the plane cuts the cone such that the plane is perpendicular to the central axis, then the primary conic generated is a circle.

**Ellipse**  If the plane cuts the cone such that

- the plane is neither perpendicular nor parallel to the axis,  
- and
- the angle of intersection is greater than the generator angle,

then the primary conic generated is an ellipse.

**Parabola**  If the plane cuts the cone such that the plane is parallel to the generator, then the primary conic generated is a parabola.

**Hyperbola**  If the plane cuts the cone such that the angle of intersection is less than the generator angle, then the primary conic generated is a hyperbola. In this case, the cutting plane intersects both nappes of the cone.
**Describing a Primary Conic by the Cutting Plane and Central Axis**

Conic sections can be determined from the angle formed by the cutting plane and the central axis of a double-napped cone.

---

Class Ex. #2

A plane intersects a double-napped cone. The primary conic produced if the plane does not pass through the vertex and cuts the cone;

a) perpendicular to the axis is ________ .

b) at an angle equal to the generator angle is ________ .

c) parallel to the axis is ________ .

d) at an angle greater than the generator angle is ________ .

e) at an angle less than the generator angle is ________ .

---

Class Ex. #3

The vertex angle of a double-napped cone is 80°. The angle between the cutting plane and the central axis is \(x\). Determine the value, or range of values, of \(x\) which would generate

a) a circle  
b) a parabola  
c) an ellipse  
d) a hyperbola

---

Class Ex. #4

In the diagram \(\alpha\) is the angle between the axis and the generator. Let \(\theta\) be the angle between the cutting plane and the axis.

For each of the primary conics, describe the relationship between \(\alpha\) and \(\theta\).

---

Class Ex. #5

Consider a cutting plane intersecting a double-napped cone at an angle just greater than the generator angle.

a) Which conic section is generated?  
b) What happens to the shape of this conic as the angle between the cutting plane and the axis increases towards 90°?

c) What happens at 90°?

---

Note

- As the cutting plane gets closer and closer to 90°, the ellipse gets more and more circular until the **limiting case of the ellipse is the circle**.
**The Primary Conics Generated from a Cylinder**

A double-napped cone is produced by rotating an oblique line (called the generator) about an axis.

**A cylinder is produced if the generator is parallel to the axis.**

---

A cylinder can also be regarded as the limiting case of a cone where the vertex is stretched to infinity.

---

Two of the primary conics (the circle and the ellipse) can be modelled from a cylinder.

**Circle**

When a plane cuts through a cylinder perpendicular to the axis, a circle is produced.

**Ellipse**

When a plane cuts through a cylinder neither perpendicular nor parallel to the axis, an ellipse is produced.

---

**Warm-Up**

Consider the situation where a plane cuts a cone perpendicular to the axis.

a) Which primary conic section is generated?

b) What happens to this conic section as the cutting plane gets closer and closer to the vertex?

c) What happens to this conic section when the cutting plane passes through the vertex?

---

**Complete Assignment Questions #1 - #13**
Extension: The Degenerate Conics from a Double-Napped Cone

In the Warm-Up we saw that as the cutting plane moves closer and closer to the vertex, the circle gets smaller and smaller until eventually, when it passes through the vertex, the circle degenerates into a point. The degenerate conics are listed below.

Point
When a plane cuts a cone at an angle greater than the generator angle and passes through the vertex, a point results. The point is a degenerate conic of a circle or an ellipse.

A Single Line
When a plane parallel to the generator passes through the vertex, a single line results. The single line is a degenerate conic of a parabola.

Two Intersecting Lines
When a plane cuts through the vertex and through both nappes of the cone, two intersecting lines result. Two intersecting lines is the degenerate conic of a hyperbola.

Note
• Notice that the degenerate conics formed from a double-napped cone only occur when the cutting plane passes through the vertex.

Extension: The Degenerate Conic from a Cylinder

Consider the intersection of a cylinder and a cutting plane parallel to the generator.

A Single Line
When the plane is tangent to the curved surface of a cylinder, a single line is produced.

Two Parallel Lines
When the plane cuts through a cylinder parallel to the axis, the result is two parallel lines.

No Graph (or No Locus)
When the plane does not intersect a cylinder, then there is no graph (or no locus).

• Since the cutting plane is parallel to the generator, the degenerate conics of a single line, two parallel lines, and no graph are regarded as degenerates of a parabola.
Assignment

1. List the primary conic sections.

2. Consider a double-napped cone whose central axis is vertical. Name the conic section produced in each case:
   a) when a plane intersects the cone parallel to the base of the cone and does not pass through the vertex.
   b) when a plane intersects both nappes of a double-napped cone and does not pass through the vertex.
   c) when a plane intersects one nappe of a double-napped cone and is parallel to the generator.
   d) when a plane intersects one nappe of a double-napped cone and is not parallel to the generator.
   e) when a plane is parallel to the axis of a double-napped cone and does not pass through the vertex.
   f) when a plane intersects a double-napped cone perpendicular to the axis of the cone and does not pass through the vertex.

3. How would you cut a double-napped cone to produce:
   a) a circle
   b) an ellipse
   c) a parabola
   d) a hyperbola

4. Which primary conic section(s) cannot be generated unless the plane intersects both nappes of a double-napped cone?

5. A cone is formed by rotating a right isosceles triangle about one of the equal sides. Which primary conic section is produced if this cone is intersected by a plane at 45° to the axis of the cone?
6. A flashlight is pointed at a wall so that the angle between the beam and the wall is 65°.
   a) Which conic section is produced?
   b) How would you adjust the angle of the beam to produce a circle on the wall?

7. A plane (not through the vertex) intersects a double-napped cone perpendicular to the axis of the cone. This plane is slowly rotated through an angle of 90° forming different conics as it rotates. In which order are the four primary conic sections produced?

8. Which of these is a limiting case of an ellipse?
   A. two parallel lines
   B. two intersecting lines
   C. a circle
   D. a single line

9. A parabola is produced by cutting a cone parallel to the generator. What happens to the parabola as the cutting plane moves closer to the vertex?
   A. it becomes wider
   B. it becomes narrower
   C. it is unchanged
   D. it becomes an ellipse

Questions 10 - 13 are based on the following information

A double-napped cone is formed by rotating a line (the generator) about a vertical axis.
The angle between the axis and the generator is 20°.
A plane intersects the double-napped cone at an angle of θ.

10. In order for the conic section produced to be a hyperbola which must be true?
    A. 0° ≤ θ < 20°
    B. θ = 20°
    C. 20° < θ < 90°
    D. θ = 90°

11. In order for the conic section produced to be a parabola which must be true?
    A. 0° ≤ θ < 20°
    B. θ = 20°
    C. 20° < θ < 90°
    D. θ = 90°

12. In order for the conic section produced to be a circle which must be true?
    A. 0° ≤ θ < 20°
    B. θ = 20°
    C. 20° < θ < 90°
    D. θ = 90°

13. In order for the conic section produced to be an ellipse which must be true?
    A. 0° ≤ θ < 20°
    B. θ = 20°
    C. 20° < θ < 90°
    D. θ = 90°
Extension Questions

14. List all the degenerate conics.

15. Consider a double-napped cone whose central axis is vertical. Name the degenerate conic section produced in each case:
   a) when a horizontal plane passes through the vertex.
   b) when an oblique plane passes through the vertex and contains the generator of a double-napped cone.
   c) when a plane is parallel to the axis of a double-napped cone and passes through the vertex.

16. How would you cut a double-napped cone to produce:
   a) a single line
   b) two intersecting lines
   c) a point

17. Which conic section(s) cannot be generated unless you consider the intersection of a plane and a cylinder?

18. Which conics (including degenerates) can be generated by the intersection of a plane and a cylinder?

19. A plane (through the vertex) intersects a double-napped cone perpendicular to the axis of the cone. This plane is slowly rotated through an angle of 90° forming different conics as it rotates. In which order are the degenerate conic sections produced?

20. Which of the following is not a degenerate conic?
   A. two parallel lines   B. two intersecting lines   C. a cylinder   D. a single line

21. The degenerate of a hyperbola is
   A. two parallel lines   B. two intersecting lines   C. a point   D. a single line
22. A degenerate of a circle or ellipse is
   A. two parallel lines  B. two intersecting lines
   C. a point            D. a single line

23. A degenerate of a parabola is
   A. a circle           B. two intersecting lines
   C. a point            D. a single line

24. If the vertex of a double-napped cone is extended infinitely, the limiting position of the
    cone is
   A. a circle           B. a line
   C. a point            D. a cylinder

**Answer Key**

1. ellipse, circle, parabola, hyperbola

2. a) circle  b) hyperbola  c) parabola  d) ellipse or circle  e) hyperbola  f) circle

3. a) perpendicular to the axis and not through the vertex
    b) through one nappe at an angle greater than the generator angle and less than 90° and not
       through the vertex
    c) parallel to the generator and not through the vertex
    d) through both nappes at an angle between 0° and the generator angle and not through the vertex

4. hyperbola  5. parabola  6. a) ellipse  b) adjust the angle to 90°

7. circle, ellipse, parabola, hyperbola


14. degenerate: point, single line, two parallel lines, two intersecting lines, no locus

15. e) point  f) line  h) two intersecting lines

16. a) parallel to the generator and through the vertex
    b) through both nappes at an angle between 0° and the generator angle and through the vertex
    c) through the vertex at an angle greater than the generator angle and less than or equal to 90°.

17 two parallel lines  18. circle, ellipse, single line, two parallel lines, no locus

19. point, single line, two intersecting lines


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Conic Sections Lesson #1: Introducing Conic Sections
Conic Sections Lesson #2: The Equation of a Conic Section in General Form

The General Form of the Equation of a Conic Section

The general form for the equation of a straight line is \( y = mx + b \).
The general form for the equation of a quadratic function is \( y = ax^2 + bx + c \).

The general form for the equation of a conic section (also called a quadratic relation) is \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \), where \( A, B, C, D, E, F \in \mathbb{R} \).

The letters \( A, B, C, D, E, \) and \( F \) are called parameters. The type of conic depends on the values of the parameters.

In this unit, we will only consider conics where \( B = 0 \) and where \( A, B, C, D, E, F \in I \)

General Form for a Quadratic Relation

\[
Ax^2 + Cy^2 + Dx + Ey + F = 0
\]

Warm-Up #1 Observations of the Parameters \( A \) and \( C \)

This warm-up will require the use of a computer with a graphing program (eg. “Zap-A-Graph”) or a graphing calculator with a conics program.

Part 1 \( A = C \)

a) Write the specified parameters in each case and sketch each graph.

<table>
<thead>
<tr>
<th>Equation 1</th>
<th>Equation 2</th>
<th>Equation 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + y^2 - 9 = 0 )</td>
<td>( x^2 + y^2 - 16 = 0 )</td>
<td>( 2x^2 + 2y^2 - 50 = 0 )</td>
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</table>

b) Complete the following statement:

In the equation \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \),

the conic produced when \( A = C \) is __________ .
**Part 2**  
*A and C both have the same sign, with \( A \neq C \), (i.e. \( AC > 0 \))

a) Write the specified parameters in each case and sketch each graph.

<table>
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<tr>
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<td>( 2x^2 + 5y^2 - 50 = 0 )</td>
<td>( -2x^2 - 5y^2 + 25 = 0 )</td>
<td>( 5x^2 + 2y^2 - 50 = 0 )</td>
<td>( -7x^2 - 3y^2 + 25 = 0 )</td>
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</table>

b) Complete the following statements:

- In the equation \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \), the conic produced when \( AC > 0 \) is __________.
- When \( |A| < |C| \) the conic is a **horizontal** __________ and has a shape like __________.
- When \( |A| > |C| \) the conic is a **vertical** __________ and has a shape like __________.

**Part 3**  
*A and C have opposite signs, (i.e. \( AC < 0 \))

a) Write the specified parameters in each case and sketch each graph.

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b) Complete the following statements:

- In the equation \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \), the conic produced when \( AC < 0 \) is __________.
- Equations ____ and ____ open along the horizontal axis.
- Equations ____ and ____ open along the vertical axis.
- If the values of \( A \) and \( C \) are interchanged, and \( F \) is not changed, then the direction of opening __________.
Part 4

A = 0, or C = 0, but NOT BOTH (i.e. AC = 0)

a) Write the specified parameters in each case and sketch each graph

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<td>$-2y^2 + 3x - 5 = 0$</td>
</tr>
<tr>
<td>$A =$</td>
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</table>

b) Complete the following statements:

- In the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, the conic produced when $AC = 0$ is __________.

- When $A = 0$ and $C \neq 0$, the conic opens __________ or __________ and has a shape like

- When $A \neq 0$ and $C = 0$, the conic opens __________ or __________ and has a shape like
General Effects of the Parameter $A$ and $C$

**Circle**

If $A = C$, then the conic is a circle.

**Ellipse**

If $A \neq C$ and they have the same sign (i.e. $AC > 0$), the conic is an ellipse.

If $|A| > |C|$ then it takes the shape $\bigcirc$

If $|A| < |C|$ then it takes the shape $\bigcirc$

The closer in value $A$ is to $C$, the closer an ellipse is to a circle.

**Hyperbola**

If $A$ and $C$ have different signs (i.e. $AC < 0$), then the conic is a hyperbola.

If $A$ and $C$ are interchanged, and $F$ remains constant, the direction of opening will change.

*The hyperbola has asymptotes which will be discussed in a later lesson.*

**Parabola**

If $A$ or $C$, not both, equal zero, then the conic is a parabola.

If $A \neq 0$ and $C = 0$,
then the parabola opens up, $\bigcup$ (eg. $y = x^2$) or down, $\bigcap$ (eg. $y = -x^2$)

If $A = 0$ and $C \neq 0$,
then the parabola opens right, $\bigcap$ (eg. $x = y^2$) or left, $\bigcup$ (eg. $x = -y^2$)

Although NOT part of the curriculum, the following two points may be of interest:

- All the conics in parts 1 to 3 of the Warm-Up have their “centre” located at the origin. This is because the parameters $D$ and $E$ are both zero. In general changing the parameters $D$ and $E$ will affect the location of the conic, left or right ($D$), or up or down ($E$), from the origin.

- Changing the parameter $F$ has a wide variety of effects, but generally involves a change in size of the conic, which may result in a degenerate conic.
State the type of conic and sketch the approximate shape of the conic represented by each of the following equations.

a) \( x^2 + 3y^2 - 6x + 8y - 90 = 0 \)

b) \( 3x^2 + 3y^2 - 4x + 5y - 63 = 0 \)

c) \( x^2 - 4x + y - 20 = 0 \)

A quadratic relation has equation \( x^2 + 2y^2 - 6x - 4y - 16 = 0 \).

a) Which type of conic is represented by the equation?

b) Determine the \( x \) and \( y \)-intercepts. Use the intercepts to make a sketch of the conic.

c) The values of \( A \) and \( C \) are interchanged.

i) How would the shape of the original conics be changed?

ii) Determine the \( x \) and \( y \)-intercepts of this new conic and make a sketch.

d) What value of \( A \) would make the original conic into a circle?
Assignment

1. Which conic is represented by each equation?

   a) \(3x^2 + 3y^2 + 12x + 4y - 54 = 0\)  
   b) \(3x^2 - 2y^2 + 7x - 14y - 57 = 0\)

   c) \(x^2 + 2y^2 - 7x + 4y - 21 = 0\)  
   d) \(x^2 + 5x - 2y - 7 = 0\)

   e) \(x^2 - y^2 - x - 8y - 50 = 0\)  
   f) \(-3x^2 + 3y^2 - 4x + 5y - 63 = 0\)

2. State the type of conic and sketch the approximate shape of the conic represented by each of the following equations.

   a) \(2x^2 + 2y^2 - 4x + 8y - 40 = 0\)  
   b) \(7x^2 + 3y^2 - 3x + 5y - 80 = 0\)

   c) \(-4x + y^2 - 20 = 0\)  
   d) \(x^2 + 3x + 5y - 21 = 0\)

   e) \(-2x^2 - 6y^2 + 8x - y + 75 = 0\)  
   f) \(x^2 + 3x - 5y - 21 = 0\)

3. Answer the following questions based on the equation \(6x^2 + 2y^2 - 9x + 14y - 68 = 0\).

   a) Which conic is represented by the equation?

   b) What value of \(A\) would transform the conic into a circle?

   c) What value of \(C\) would transform the original conic into a circle?

   d) What change would take place if the values of \(A\) and \(C\) were interchanged?
4. Consider the equation \( x^2 + y^2 - 6x + 10y + 9 = 0 \)
   a) Which quadratic relation is represented by the equation?

   b) Susan used a computer program to graph the quadratic relation. The resulting graph was a hyperbola. When she checked the equation on her computer, she found she had entered one of the signs incorrectly. Which of the four signs was incorrectly entered?

   c) Determine the \( x \) and \( y \)-intercepts of the original equation and sketch the graph.

5. Answer the following questions given the equation \( 3x^2 - y^2 - 36x + y + 60 = 0 \)
   a) What type of curve does this equation represent?

   b) Determine the \( x \) and \( y \)-intercepts of the original equation and sketch the graph.

   c) If the values of \( A \) and \( C \) are interchanged in the equation, what effect will this have on the basic shape of the graph.
6. Which equation represents an ellipse?

A. \(-2x^2 - 2y^2 - 100 = 0\)
B. \(-2x^2 + 2y^2 + 100 = 0\)
C. \(x^2 - 2y - 100 = 0\)
D. \(-x^2 - 2y^2 + 100 = 0\)

7. Which equation represents a hyperbola?

A. \(-2x^2 + 2y^2 - 100 = 0\)
B. \(2x^2 + 2y^2 + 100 = 0\)
C. \(-x^2 - 2y^2 - 100 = 0\)
D. \(-x^2 - 2y^2 + 100 = 0\)

8. Which equation represents a non-degenerate parabola?

A. \(2y^2 - 3x + 10 = 0\)
B. \(2x^2 - 3x + 10 = 0\)
C. \(x^2 - 2y^2 - 3x - 10 = 0\)
D. \(x^2 - 2y^2 - 3y + 10 = 0\)

9. In the equation
\(Ax^2 + Cy^2 + 8x + 10y - 34 = 0\)
and either \(A\) or \(C\) = 0, then the curve is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

10. In the equation
\(Ax^2 + Bxy + Cy^2 + 8x + 10y - 34 = 0\)
if \(B = 0\) and \(A, C > 0, A = C\), then the curve is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

11. In the equation
\(Ax^2 + Cy^2 + 6x - 10y + 40 = 0, AC < 0, A \neq C\), then the curve is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

12. In the equation
\(Ax^2 + Cy^2 + 6x - 10y + 40 = 0, A, C < 0, A \neq C\), then the curve is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola

13. If \(A = 0\) and \(C = 2\) in the equation
\(Ax^2 + Cy^2 + 8x + 10y - 34 = 0\)
then the curve is a parabola opening

A. left
B. right
C. down
D. up

14. The equation
\(Ax^2 + Cy^2 + Dx + Ey + F = 0\)
represents a hyperbola if

A. \(AC > 0, A \neq C\)
B. \(AC < 0, A \neq C\)
C. \(AC = 0\)
D. \(A = C\)

15. The conic given by
\(Ax^2 - 2y^2 + 14 = 0\) with \(A < 0\) and \(A \neq -2\) is

A. a circle
B. a parabola
C. an ellipse
D. a hyperbola
16. If $Ax^2 + Cy^2 - 1 = 0$ represents a circle and $A = 10$, then
   - A. $C = 0$
   - B. $C = 10$
   - C. $C > 10$
   - D. $C < 10$

17. Which equation represents a vertical ellipse?
   - A. $-4x^2 - 2y^2 + 100 = 0$
   - B. $x^2 + 2y^2 - 100 = 0$
   - C. $-x^2 + 2y^2 - 100 = 0$
   - D. $x^2 - 2y^2 + 100 = 0$

18. A quadratic relation is defined by $Ax^2 + Cy^2 + Dx + Ey + F = 0$.
   If none of the parameters are zero the only shape that is not possible is
   - A. a circle
   - B. a hyperbola
   - C. an ellipse
   - D. a parabola

19. The equation $2x^2 + 5y^2 - 10y + 40 = 0$ represents a conic section formed by a plane intersecting a double-napped cone at an angle
   - A. equal to the generator angle
   - B. greater than the generator angle
   - C. less than the generator angle
   - D. perpendicular to the axis

---

Use the following information to answer the next question.

The graph of the quadratic relation $2x^2 + y^2 - 3x - 2y - 25 = 0$ is shown.

20. The resulting graph when the coefficients $A$ and $C$ are interchanged is
   - A.
   - B.
   - C.
   - D.
21. In the equation \( Ax^2 + Cy^2 + F = 0 \) if \( B = 0 \) and \( A, C > 0, A < C \), then the curve is
A. an ellipse with the longer axis along the \( x \)-axis.
B. an ellipse with the longer axis along the \( y \)-axis.
C. a hyperbola along the \( x \)-axis.
D. a hyperbola along the \( y \)-axis.

Numerical Response 22. The positive \( y \)-intercept, to the nearest hundredth, of the parabola \( y^2 - 2x + 3y - 7 = 0 \) is ______ .

**Answer Key**
1. a) circle  b) hyperbola  c) ellipse  d) parabola  e) hyperbola  f) hyperbola
2. a) circle  b) ellipse  c) parabola  d) parabola  e) ellipse  f) parabola
3. a) ellipse  b) 2  c) 6  d) ellipse would have its longer axis parallel to the \( x \)-axis (i.e. horizontal ellipse).
4. a) circle  b) the + sign in front of the \( y^2 \) was entered as a – sign.
   c) \( x \)-intercept = 3, \( y \)-intercepts = -9 and -1
5. a) hyperbola  b) \( x \)-intercepts = 2 and 10, \( y \)-intercepts = \( \frac{1 \pm \sqrt{241}}{2} \) = -7.26 and 8.26
   c) hyperbola would open along the vertical axis
20. B  21. A  22. 1.54
Conic Sections Lesson #3: 
The Equation of a Conic Section in Standard Form

**Warm-Up #1**

The four equations below represent the equations of different conic sections, but they are in a format we are not yet familiar with.

i) \( x - 3 = 2(y + 3)^2 \)  
ii) \( (x - 2)^2 + (y + 1)^2 = 4 \)

iii) \( \frac{x^2}{9} + \frac{(y + 1)^2}{25} = 1 \)  
iv) \( \frac{(x - 5)^2}{4} - \frac{(y - 2)^2}{16} = 1 \)

- Convert each equation into the general form \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \) and state which type of conic section each equation represents.

**Warm-Up #2**

The equations in Warm-Up #1 are examples of conic sections whose equations are written in standard form.

Use a graphing program such as Zap-a-Graph to sketch the conics from Warm-Up #1. Use general form and standard form to verify the graphs are identical.
The Standard Form of the Equation of a Conic Section

The equations listed below are the **standard form** of the equation of each type of conic section.

### Parabola

- Opening up or down
  \[ y - k = a(x - h)^2 \]

- Opening left or right
  \[ x - h = a(y - k)^2 \]

### Circle

\[(x - h)^2 + (y - k)^2 = r^2 \]

or
\[ \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \quad \text{where } a = r \]

### Ellipse

\[ \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \]

- For a horizontal ellipse, \(a^2 > b^2\).
- For a vertical ellipse, \(a^2 < b^2\).

### Hyperbola

- Opening along the x-axis
  \[ \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \]

- Opening along the y-axis
  \[ \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = -1 \]

or
\[ \frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1 \]

- The hyperbola has asymptotes with slopes equal to \( \pm \frac{b}{a} \).

The above information is **NOT** on the formula sheet.
See the formula below for the standard form of equations of conics.

### Standard Form for a Quadratic Relation

\[ \frac{(x - h)^2}{a^2} \pm \frac{(y - k)^2}{b^2} = \pm 1 \]

\[ y - k = a(x - h)^2 \]

\[ x - h = a(y - k)^2 \]

*These formulas are **NOT** on the formula sheet*
In the next lesson we will use the standard form to identify features of the graph of a conic section such as centre, intercepts, domain, and range.

Identify the type of conic section from the equation. Do not use technology.

a) \( x^2 + y^2 = 16 \)  

b) \( x + 5 = \frac{1}{2} (y - 3)^2 \)  

c) \( \frac{(x - 4)^2}{4} - y^2 = 1 \)  

**Complete Assignment Questions #1 - #2**

**Assignment**

1. State the type of conic section which each equation represents.

   a) \( (x - 2)^2 + (y - 4)^2 = 64 \)  
   
   b) \( y - 2 = 5x^2 \)  
   
   c) \( \frac{(x - 7)^2}{8} + \frac{(y + 2)^2}{20} = 1 \)  
   
   d) \( (x - 2)^2 - (y - 4)^2 = 64 \)  
   
   e) \( x - 2 = 4(y - 3)^2 \)  
   
   f) \( \frac{(y - 7)^2}{8} - \frac{(x + 2)^2}{20} = 1 \)  

2. For each of the following quadratic relations defined in standard form;

   i) State the type of conic section represented
   
   ii) Convert the equation into general form.

   a) \( \frac{(x - 3)^2}{4} + \frac{(y + 2)^2}{4} = 1 \)  
   
   b) \( \frac{x^2}{4} - \frac{(y + 5)^2}{9} = 1 \)
c) \( y + 1 = 4(x - 6)^2 \)

d) \( \frac{(x + 4)^2}{9} + \frac{(y - 4)^2}{36} = 1 \)

**Answer Key**

1. a) circle  b) parabola  c) ellipse  d) hyperbola  e) parabola  f) hyperbola

2. a) i) circle  ii) \( x^2 + y^2 - 6x + 4y + 9 = 0 \)
   b) i) hyperbola  ii) \( 9x^2 - 4y^2 - 40y - 136 = 0 \)
   c) i) parabola  ii) \( 4x^2 - 48x - y + 143 = 0 \)
   d) i) ellipse  ii) \( 4x^2 + y^2 + 32x - 8y + 44 = 0 \)
Conic Sections Lesson #4: Transformations of Parabolas

Review of Transformations

Recall the following transformations which are relevant to this unit.

<table>
<thead>
<tr>
<th>Replacement for x or y</th>
<th>Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \rightarrow \frac{1}{a}x$</td>
<td>horizontal stretch by a factor of $a$ about the y-axis</td>
</tr>
<tr>
<td>$y \rightarrow \frac{1}{b}y$</td>
<td>vertical stretch by a factor of $b$ about the x-axis</td>
</tr>
<tr>
<td>$x \rightarrow x - h$</td>
<td>horizontal translation $h$ units right</td>
</tr>
<tr>
<td>$y \rightarrow y - k$</td>
<td>vertical translation $k$ units up</td>
</tr>
<tr>
<td>$x \rightarrow -x$</td>
<td>reflection in the y-axis</td>
</tr>
<tr>
<td>$y \rightarrow -y$</td>
<td>reflection in the x-axis</td>
</tr>
</tbody>
</table>

Describe how the graph of the second relation compares to the graph of the first relation.

a) $y = x^2$
   
   $y - 3 = x^2$

b) $x^2 + y^2 = 1$
   
   $(x - 2)^2 + (y + 3)^2 = 1$

   $\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{5}y\right)^2 = 1$
**Transformations of the Parabola**

The graph of the parabola with equation \( y = x^2 \) is shown. Complete the following for the graph.

- domain _______________
- range _______________
- coordinates of vertex ______

**Part 1  Transforming the Parabola using a Numerical Value for the Stretch Factor**

\( x \) is replaced by \( \frac{1}{2} x \) to get the equation \( y = \left(\frac{1}{2} x\right)^2 \) or \( y = \frac{1}{4} x^2 \).

a) Complete the following for the graph of the equation \( y = \frac{1}{4} x^2 \).
   - The transformation from \( y = x^2 \) is a horizontal ___________ by a factor of _____ about the \( y \)-axis.

b) Draw the transformed image on the grid and complete.
   - domain _______________
   - range _______________
   - coordinates of vertex ______

**Part 2A  Transforming the Parabola using an Algebraic Value for the Stretch Factor**

\( x \) is replaced by \( \sqrt{a} x \) (where \( a > 0 \)) to get the equation \( y = \left(\sqrt{a} x\right)^2 \) or \( y = ax^2 \).

a) Complete the following for the graph of the equation \( y = ax^2 \) where \( a > 0 \).
   - The transformation from \( y = x^2 \) is a horizontal stretch by a factor of _____ about the \( y \)-axis

b) Draw the transformed image on the grid and complete.
   - domain _______________
   - range _______________
   - coordinates of vertex ______
Part 2B  Transforming the Parabola with a Stretch and Translations

• $x$ is then replaced by $x - h$ and $y$ is then replaced by $y - k$ to get the equation $y - k = a(x - h)^2$, where $a > 0$.

a) Complete for the graph of the equation $y - k = a(x - h)^2$.

• The transformation from $y = ax^2$ is

b) Label the transformed image on the grid and complete.

• domain __________________
• range ________________
• coordinates of vertex ______

c) What changes, if any, would there be to the answers to b) if the equation was in the form $y - k = a(x - h)^2$, where $a < 0$?

Features of the Graph of the Parabola  $y - k = a(x - h)^2$

The parabola defined by the equation $y - k = a(x - h)^2$ has

• vertex $(h, k)$
• domain $x \in \mathbb{R}$.
• If $a > 0$ the range is $y \geq k$ and if $a < 0$ the range is $y \leq k$.
• $x$- and $y$-intercepts are determined by solving the equations $y = 0$ and $x = 0$ respectively.

Note

Compared to the graph of $y = x^2$ the graph of $y = ax^2$, $a > 0$, can be regarded as either

• a horizontal stretch by a factor of $\frac{1}{\sqrt{a}}$ about the $y$-axis or
• a vertical stretch by a factor of $a$ about the $x$-axis.

In this lesson, we will use the horizontal stretch as it will help us with transformations of circles, ellipses, and hyperbolas.
Class Ex. #2

Consider the conic section with equation \( y - 9 = -9(x - 2)^2 \).

a) Describe the series of transformations which would transform the graph \( y = x^2 \) to the graph of \( y - 9 = -9(x - 2)^2 \).

b) Determine the following features of the graph of \( y - 9 = -9(x - 2)^2 \).

i) domain

ii) range

iii) vertex

iv) axis of symmetry

v) \( x \)- and \( y \)-intercepts

c) Use the information above to sketch the graph of \( y - 9 = -9(x - 2)^2 \).

Class Ex. #3

The graph shown has equation \( y - k = -(x - h)^2 \).
The vertex has coordinates \( V(-3, 4) \).

a) Determine the values of \( h \) and \( k \).

b) Write the equation in general form.

c) Find the \( x \)- and \( y \)-intercepts.

Complete Assignment Questions #1 - #3
**Transformations of the Parabola** $x = y^2$

The graph of the parabola with equation $x = y^2$ is shown. Complete the following for the graph.

- domain _______________
- range _______________
- coordinates of vertex ______

**Part 1**  
*Transforming the Parabola using a Numerical Value for the Stretch Factor*

$y$ is replaced by $\frac{1}{3}y$ to get the equation $x = \left(\frac{1}{3}y\right)^2$ or $x = \frac{1}{9}y^2$.

a) Complete the following for the graph of the equation $x = \frac{1}{9}y^2$.
- The transformation from $x = y^2$ is a vertical _______________ by a factor of _____ about the $x$-axis.

b) Draw the transformed image on the grid and complete.
- domain _______________
- range _______________
- coordinates of vertex ______

**Part 2**  
*Transforming the Parabola with a Stretch and Translations*

$x$ is then replaced by $x - 3$ and $y$ by $y - 4$ to get the equation $x - 3 = \frac{1}{9}(y - 4)^2$.

a) Complete for the graph of the equation $x - 3 = \frac{1}{9}(y - 4)^2$.
- The transformation from $x = \frac{1}{9}y^2$ is

b) Label the transformed image on the grid and complete.
- domain _______________  
- range _______________
- coordinates of vertex ______

c) What changes, if any, would there be to the answers to b) if the equation was $x - 3 = -\frac{1}{9}(y - 4)^2$?
Features of the Graph of the Parabola \( x - h = a(y - k)^2 \)

The parabola defined by the equation \( x - h = a(y - k)^2 \) has

- vertex \((h, k)\)
- range \( y \in \mathbb{R} \).
- If \( a > 0 \) the domain is \( x \geq h \) and if \( a < 0 \) the domain is \( x \leq h \).
- \( x \)- and \( y \)-intercepts are determined by solving the equations \( y = 0 \) and \( x = 0 \) respectively.

**Note**

Compared to the graph of \( x = y^2 \), the graph of \( x = ay^2, a > 0 \), can be regarded as either

- a vertical stretch by a factor of \( \frac{1}{\sqrt{a}} \) about the \( x \)-axis or
- a horizontal stretch by a factor of \( a \) about the \( y \)-axis.

In this lesson, we will use the vertical stretch as it will help us with transformations of circles, ellipses, and hyperbolas.

Class Ex. #4

Consider the conic section with equation \( x - 3 = \frac{1}{16} (y + 4)^2 \).  

a) Describe the series of transformations which would transform the graph \( x = y^2 \) to the graph of \( x - 3 = \frac{1}{16} (y + 4)^2 \).

b) Determine the following features of the graph of \( x - 3 = \frac{1}{16} (y + 4)^2 \).

i) domain  
ii) range  
iii) coordinates of the vertex  
iv) \( x \)- and \( y \)-intercepts

c) Use the information above to sketch the graph of \( x - 3 = \frac{1}{16} (y + 4)^2 \).
A graph has an equation of the form $x - h = -(y - k)^2$. The vertex has coordinates $(-1, 3)$.

a) State the equation of the graph.

Extension

b) Determine the equation of the parabola if it is stretched horizontally by a factor of 2 about the line $x = 1$.

---

**Assignment**

1. Determine the equation of the parabola $y = x^2$ after each of the following transformations:
   a) translated 3 units down
   b) horizontal translation 5 units right
   c) horizontal expansion by a factor of 4 about the line $x = 0$
   d) vertical compression by a factor of $\frac{1}{9}$ about the line $y = 0$. 
2. Consider the conic section with equation \( y + 9 = \frac{1}{4}(x - 4)^2 \).

a) Use transformations to describe how the graph of this conic section compares to the graph of the parabola \( y = x^2 \).

b) Determine the following features of the graph of \( y + 9 = \frac{1}{4}(x - 4)^2 \).
   i) domain  
   ii) range  
   iii) vertex  
   iv) axis of symmetry  
   v) \( x \)- and \( y \)-intercepts

c) Use the information above to sketch the graph of \( y + 9 = \frac{1}{4}(x - 4)^2 \).

3. Consider the conic section with equation \( y = -2(x + 6)^2 \).

a) Use transformations to describe how the graph of this conic section compares to the graph of the parabola \( y = x^2 \).

b) Determine the following features of the graph of \( y = -2(x + 6)^2 \).
   i) domain
   ii) range
   iii) coordinates of the vertex
   iv) \( x \)- and \( y \)-intercepts

c) Use the information above to sketch the graph of \( y = -2(x + 6)^2 \).
4. Consider the conic section with equation $x - 1 = -4(y - 2)^2$.
   a) Use transformations to describe how the graph of this conic section compares to the graph of the parabola $x = y^2$.

   b) Determine the following features of the graph of $x - 1 = -4(y - 2)^2$.
      i) domain
      ii) range
      iii) vertex
      iv) axis of symmetry
      v) $x$- and $y$-intercepts

   c) Use the information above to sketch the graph of $x - 1 = -4(y - 2)^2$.

5. The graph shown is a transformed image of the graph of $y = x^2$. The transformation consists of a horizontal expansion by a factor of 2 about the $y$-axis, followed by a translation.
   a) Determine the equation of the graph in general form.

   ![Graph](image)

   **Extension**

   b) The given graph is compressed vertically by a factor of $\frac{1}{2}$ about the line $y = -4$. Determine the equation of the transformed graph in standard form.
6. The graph shown is a transformed image of the graph of \( x = y^2 \). The transformation consists of a horizontal expansion by a factor of 2 about the \( y \)-axis, followed by a translation. Determine the equation of the graph in general form.

7. A graph has an equation of the form \( x - h = 3(y - k)^2 \). The vertex is \( V(5, 0) \). If the graph is translated 3 units left and 2 units up, determine the equation of the transformed image in standard form.

8. The graph shows a parabola with equation \( x - h = a(y - k)^2 \). How many of the parameters \( a, h, \) and \( k \) are positive?

A. 0  
B. 1  
C. 2  
D. 3

9. The graph shown represents an equation of the form \( y - k = a(x - h)^2 \). If the vertex is \( V(1, 3) \) and the graph passes through the point \( P(2, 7) \), the value of \( a \), to the nearest tenth is ______ .

Answer Key
1. a) \( y = x^2 - 3 \)  b) \( y = (x - 5)^2 \)  c) \( y = \frac{1}{16}x^2 \)  d) \( y = \frac{1}{9}x^2 \)
2. a) horizontal expansion by a factor of \( \sqrt{2} \) about the \( y \)-axis, then a translation 4 units right and 9 units down  
b) i) \( x \in \mathbb{R} \)  ii) \( \{y \mid y \geq -9, y \in \mathbb{R}\} \)  iii) \( (4, -9) \)  iv) \( x = 4 \)  v) \( x\text{-int} = -2 \) and \( 10, y\text{-int} = -5 \)
3. a) horizontal compression by a factor of \( \frac{\sqrt{2}}{2} \) about the \( x \)-axis, reflection in the \( y \)-axis, then a translation 6 units left.  
b) i) \( x \in \mathbb{R} \)  ii) \( \{y \mid y \leq 0, y \in \mathbb{R}\} \)  iii) \( (-6, 0) \)  iv) \( x\text{-int} = -6, y\text{-int} = -72 \)
4. a) vertical compression by a factor of \( \frac{1}{2} \) about the \( x \)-axis, reflection in the \( y \)-axis, then a translation 1 unit right and 2 units up  
b) i) \( \{x \mid x \leq 1, x \in \mathbb{R}\} \)  ii) \( y \in \mathbb{R} \)  iii) \( (1, 2) \)  iv) \( y = 2 \)  v) \( x\text{-int} = -15, y\text{-int} = 1.5 \) and \( 2.5 \)
5. a) \( x^2 - 8x - 4y + 8 = 0 \)  b) \( y + 3 = \sqrt[3]{x - 4} \)
6. \( 2y^2 - x - 12y + 11 = 0 \)  7. \( x - 2 = 3(y - 2)^2 \)  8. A  9. 4.0
Conic Sections Lesson #5: Transformations of Circles and Ellipses

Transitions of the Circle $x^2 + y^2 = 1$

The graph of the circle with equation $x^2 + y^2 = 1$ is shown. Complete the following for the graph.

- domain ____________  • range ____________
- centre ______  • radius ______

Part 1  Transforming the Circle using Stretches

$x$ is replaced by $\frac{1}{2}x$ and $y$ is replaced by $\frac{1}{2}y$ to get the equation

$$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{2}y\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{2^2} + \frac{y^2}{2^2} = 1 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{4} = 1 \quad \text{or} \quad x^2 + y^2 = 4.$$

a) Complete the following for the graph of the equation $\frac{x^2}{4} + \frac{y^2}{4} = 1$.
- The transformation from $x^2 + y^2 = 1$ is

b) Draw the transformed image on the grid and complete.
- domain____________  • range____________
- centre ______  • radius ______

Part 2A  Transforming the Circle using Stretches

$x$ is replaced by $\frac{1}{2}x$ and $y$ is replaced by $\frac{1}{3}y$ to get the equation

$$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{3}y\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

a) Complete for the graph of the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
- The transformation from $x^2 + y^2 = 1$ is

b) Draw the transformed image and label the intercepts.
- domain____________  • range ____________  • centre ______
- length of horizontal axis (horizontal diameter) ______
- length of vertical axis (vertical diameter) ______
Part 2B

Transforming the Ellipse with Translations

The ellipse in Part 2A is then transformed as follows:

\[
x \text{ is then replaced by } x - 4 \text{ and } y \text{ is then replaced by } y - 1 \text{ to get the equation}
\]

\[
\frac{(x - 4)^2}{4} + \frac{(y - 1)^2}{9} = 1.
\]

a) Complete for the graph of the equation \(\frac{(x - 4)^2}{4} + \frac{(y - 1)^2}{9} = 1\).

- The transformation from \(\frac{x^2}{4} + \frac{y^2}{9} = 1\) is

b) Label the transformed image on the grid and complete.

- domain ____________
- range ____________
- centre _____
- vertices ____________
- length of major axis _____
- length of minor axis _____

Part 3

The General Case

The circle \(x^2 + y^2 = 1\) is transformed into the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) by a horizontal stretch about the \(y\)-axis by a factor of ___ and a vertical stretch about the \(x\)-axis by a factor of ___.

The ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) is transformed into the ellipse \(\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1\) by a translation ___ units right and ___ units up.

The diagram shows the circle and both ellipses

On the diagram label the coordinates of each point shown.

Complete for \(\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1\).

- domain _________________
- range _________________
- coordinates of centre _____
- length of horizontal axis _____
- length of vertical axis _____
Features of the Graph of the Ellipse \[ \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \]

The ellipse defined by the equation \[ \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \] has

- centre \((h, k)\)
- domain \(-a + h \leq x \leq a + h\)
- range \(-b + k \leq y \leq b + k\)
- the length of the horizontal axis of \(2a\)
- the length of the vertical axis length of \(2b\)
- the longer of the two axes is called the major axis, the shorter is called the minor axis
- \(a\) is the horizontal stretch factor and \(b\) is the vertical stretch factor from the unit circle
- \(h\) is the horizontal translation and \(k\) is the vertical translation of the centre.

The circle can be considered as a special case of the ellipse with \(b = a\).

Features of the Graph of the Circle \[ \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2} = 1 \]

or

\[ (x - h)^2 + (y - k)^2 = a^2 \]

The circle defined by the equation \[ \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2} = 1 \] has

- centre \((h, k)\)
- radius \(a\)
- domain \(-a + h \leq x \leq a + h\)
- range \(-a + k \leq y \leq a + k\)
Consider the ellipse with equation \( \frac{(x + 2)^2}{25} + \frac{(y - 4)^2}{9} = 1 \).

a) Use transformations to describe how the graph of this conic section compares to the graph of the circle \( x^2 + y^2 = 1 \).

b) Determine the following features of the graph of \( \frac{(x + 2)^2}{25} + \frac{(y - 4)^2}{9} = 1 \).
   i) centre   ii) horizontal length   iii) vertical length
   iv) domain   v) range   vi) coordinates of vertices

c) Use the information above to sketch the graph of \( \frac{(x + 2)^2}{25} + \frac{(y - 4)^2}{9} = 1 \).

The graph of a circle with centre (0, 0) and radius 1 is transformed into an ellipse as shown.

a) Describe the series of transformations required to transform the graph of the circle into the graph of an ellipse with centre at (4, -6), a horizontal length of 4 units and a vertical length of 8 units.

b) Write the equation of the transformed image.
Consider the circle with equation \[ \frac{(x - 3)^2}{36} + \frac{(y + 4)^2}{36} = 1. \]

a) Describe the series of transformations which would transform the graph \( x^2 + y^2 = 1 \) to the graph of \[ \frac{(x - 3)^2}{36} + \frac{(y + 4)^2}{36} = 1. \]

b) Determine the following features of the graph of \[ \frac{(x - 3)^2}{36} + \frac{(y + 4)^2}{36} = 1. \]
   i) centre
   ii) radius
   iii) domain
   iv) range

c) Use the information above to sketch the graph of \[ \frac{(x - 3)^2}{36} + \frac{(y + 4)^2}{36} = 1. \]

In each case, write the equation of the circle in the form \( (x - h)^2 + (y - k)^2 = r^2 \) and in the form \[ \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2} = 1. \]

a) centre \((3, -7)\) and diameter 10

b) endpoints of a diameter at \((4, 2)\) and \((8, 2)\)
A translation of $p$ units right and $q$ units up can be described by the ordered pair $(p, q)$.

a) Determine the equation of the circle \( \frac{x^2}{100} + \frac{y^2}{100} = 1 \) after a translation described by the ordered pair $(5, -2)$.

b) If the point $P(6, 8)$ lies on the original circle, determine the coordinates of $P'$, the image of $P$, under the transformation in a).

c) If a point $Q(m, n)$ lies on the original circle, determine the coordinates of $Q'$, the image of $Q$, under the transformation in a).

**Complete Assignment Questions #1 - #17**

**Assignment**

1. Determine the equation of the given conic after the transformation.

   a) Determine the equation of the image of the circle $(x - 2)^2 + (y - 3)^2 = 1$ after a translation 2 units up and 3 units left.

   b) Determine the equation of the image of the ellipse \( \frac{x^2}{25} + \frac{y^2}{4} = 1 \) after a horizontal expansion of factor 2 about the $y$-axis and a vertical compression of factor \( \frac{1}{2} \) about the $x$-axis.

   c) Determine the equation of the image of the circle $(x - 2)^2 + (y + 5)^2 = 20$ under the translation defined by the mapping $(x, y) \rightarrow (x - 2, y + 5)$. 
2. Use transformations to describe how the graph of the given conic can be obtained from the graph of the circle \( x^2 + y^2 = 1 \).

   a) \( \frac{x^2}{16} + \frac{(y + 4)^2}{9} = 1 \)

   b) \( 4x^2 + 4y^2 = 1 \)

   c) \( \frac{25(x - 1)^2}{16} + \frac{(y + 3)^2}{121} = 1 \)

   d) \( 9x^2 + 9(y - 8)^2 = 4 \)

3. A translation of \( p \) units right and \( q \) units up can be described by the ordered pair \( (p, q) \).

   a) Determine the equation of the circle \( (x - 2)^2 + y^2 = 25 \) after a translation described by the ordered pair \((-3, 4)\).

   b) If the point \( P(5, 4) \) lies on the original circle, determine the coordinates of \( P' \), the image of \( P \), under the transformation in a).

   c) If a point \( Q(m, n) \) lies on the original circle, determine the coordinates of \( Q' \), the image of \( Q \), under the transformation in a).
4. Consider the ellipse with equation \((x + 2)^2 + \frac{(y + 1)^2}{4} = 1\).

   a) Use transformations to describe how the graph of this conic section compares to the graph of the circle \(x^2 + y^2 = 1\).

   b) Determine the following features of the graph of \((x + 2)^2 + \frac{(y + 1)^2}{4} = 1\).
      i) centre  ii) length of minor axis  iii) length of major axis
      iv) domain  v) range  vi) coordinates of vertices

   c) Use the information above to sketch the graph of \((x + 2)^2 + \frac{(y + 1)^2}{4} = 1\).

5. Consider the circle with equation \(x^2 + (y - 12)^2 = 81\).
   a) Describe the series of transformations which would transform the graph \(x^2 + y^2 = 1\) to the graph of \(x^2 + (y - 12)^2 = 81\).

   b) Determine the following features of the graph of \(x^2 + (y - 12)^2 = 81\).
      i) centre  ii) radius  iii) domain  iv) range
      v) \(x\)- and \(y\)-intercepts
c) Use the information in a) and b) to sketch the graph of \( x^2 + (y - 12)^2 = 81. \)

d) The graph of \( x^2 + (y - 12)^2 = 81 \) is compressed horizontally by a factor of \( \frac{2}{3} \) about the line \( x = 6. \)

Sketch the transformed graph and determine its equation.

6. In each case, write the equation of the circle in the form \( (x - h)^2 + (y - k)^2 = r^2. \)

a) centre \((-8, 11)\) and radius \( \frac{5}{3} \)  

b) endpoints of a diameter at \((3, 3)\) and \((5, 5)\)

7. In each case, write the equation of the circle in the form \( \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2} = 1. \)

a) centre \((6, -1)\) and passing through \((0, 7)\)  

b) centre \((5, 4)\) and tangent to the \(x\)-axis.
8. Consider the following ellipses

In each case describe the transformations that have been applied to the unit circle $x^2 + y^2 = 1$ to produce the ellipse. Express the equation of each ellipse in standard form.
9. Determine the equation of each of the following ellipses.
   a) Vertices are (0, –6) and (0, 6) and horizontal length is 8 units.
   b) Vertices are (–10, 3) and (2, 3) and vertical length is 4 units.

10. a) Sketch the graph of the circle with equation \(x^2 + y^2 = 9\).

Extension
   b) Sketch the graph after a vertical expansion by a factor of 2 about the line \(y = -3\).
   c) Determine the equation of the transformed graph.

11. a) Determine the equation of an ellipse with centre (1, –2), horizontal length of 6 units, and vertical length of 4 units.

Extension
   b) The ellipse in a) undergoes a horizontal compression by a factor of \(\frac{1}{3}\) about the line \(x = -5\). Determine the equation of the transformed ellipse.
12. Describe the series of transformations which would transform the graph $x^2 + y^2 = 4$ to the given graph and write the equation of the given graph in general form.

13. Which of the following circles has a diameter of $\sqrt{20}$?

   - A. $(x - 2)^2 + (y - 2)^2 = \sqrt{5}$
   - B. $4x^2 + 4y^2 = 20$
   - C. $\frac{x^2}{4} + \frac{(y - 3)^2}{4} = \frac{5}{2}$
   - D. $x^2 = 20 - y^2$

14. The diagram shows the ellipse with equation $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$.
   
   The value of $a + b$ is
   
   - A. 20
   - B. 12
   - C. 6
   - D. $-1$
15. Consider the ellipse with equation \( \frac{(x - 8)^2}{100} + \frac{(y + 1)^2}{64} = 1 \). Which one of the following statements is false?

A. The ellipse contains points in all four quadrants.
B. The line \( x = 8 \) is an axis of symmetry.
C. The centre lies in quadrant 4.
D. The ellipse is a vertical ellipse.

16. Consider the quadratic relation with equation \( \frac{(x - 3)^2}{81} + \frac{4(y + 2)^2}{49} = 1 \).

The maximum \( y \) coordinate, to the nearest tenth, on the graph of the relation is _____.

17. The circle of the equation \( 5x^2 + 5y^2 = 1 \) is a transformed image of the circle with equation \( x^2 + y^2 = 1 \). The scale factor, to the nearest hundredth, of the horizontal and vertical stretch is _____.

**Answer Key**

1. a) \( (x + 1)^2 + (y - 5)^2 = 1 \)  
   b) \( \frac{x^2}{100} + y^2 = 1 \)  
   c) \( x^2 + y^2 = 20 \)

2. a) a horizontal expansion by a factor of 4 about the \( y \)-axis, a vertical expansion by a factor of 3 about the \( x \)-axis, followed by a translation 4 units down.
   
   b) horizontal compression about the \( y \)-axis and vertical compression about the \( x \)-axis by a factor of \( \frac{1}{2} \).
   
   c) a horizontal compression by a factor of \( \frac{4}{5} \) about the \( y \)-axis, a vertical expansion by a factor of 11 about the \( x \)-axis, followed by a translation 1 unit right and 3 units down.
   
   d) horizontal compression about the \( y \)-axis and vertical compression about the \( x \)-axis by a factor of \( \frac{2}{3} \) followed by a vertical translation 8 units up.

3. a) \( (x + 1)^2 + (y - 4)^2 = 25 \)  
   b) \( (2, 8) \)  
   c) \( (m - 3, n + 4) \)

4. a) vertical expansion by a factor of 2 about the \( x \)-axis, followed by a translation 2 units left and 1 unit down
   
   b) i) \( (-2, -1) \)  
   ii) 2  
   iii) 4  
   iv) \( \{ x \mid -3 \leq x \leq -1, x \in \mathbb{R} \} \)
   
   v) \( \{ y \mid -3 \leq y \leq 1, y \in \mathbb{R} \} \)  
   vi) \( (-2, -3) \) and \( (-2, 1) \)
5. a) horizontal expansion about the y-axis by a factor of 9 and vertical expansion about the x-axis by a factor of 9, followed by a translation 12 units up.

b) i) (0, 12) ii) 9 iii) \{x \,|\, -9 \leq x \leq 9, x \in \mathbb{R}\} iv) \{y \,|\, 3 \leq y \leq 21, y \in \mathbb{R}\} v) x\text{-int} = \text{none}, y\text{-int} = 3 \text{ and } 21

c) see graph below
d) \[ \frac{(x-2)^2}{36} + \frac{(y-12)^2}{81} = 1 \]

6. a) \( (x+8)^2 + (y-11)^2 = \frac{25}{9} \)

7. a) \[ \frac{(x-6)^2}{100} + \frac{(y+1)^2}{100} = 1 \]

b) \( \frac{(x-5)^2}{16} + \frac{(y-4)^2}{16} = 1 \)

8. **ellipse 1** a horizontal expansion by a factor of 5 about the y-axis and 
a vertical expansion by a factor of 3 about the x-axis: \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \)

**ellipse 2** a horizontal expansion by a factor of 2 about the y-axis, 
a vertical expansion by a factor of 3 about the x-axis, 
followed by a horizontal translation 8 units left: \( \frac{(x+8)^2}{4} + \frac{y^2}{9} = 1 \)

**ellipse 3** a horizontal expansion by a factor of 4 about the y-axis, a vertical expansion by a factor of 8 about the x-axis, then a translation 10 units right and 4 units up: \( \frac{(x-10)^2}{16} + \frac{(y-4)^2}{64} = 1 \)

**ellipse 4** a horizontal expansion by a factor of 6 about the y-axis, a vertical expansion by a factor of 2 about the x-axis, then a translation 5 units left and 8 units up: \( \frac{(x+5)^2}{36} + \frac{(y-8)^2}{4} = 1 \)

9. a) \( \frac{x^2}{16} + \frac{y^2}{36} = 1 \)

b) \( \frac{(x+4)^2}{36} + \frac{(y-3)^2}{4} = 1 \)

10. a) and b) see graph below

11. a) \( \frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1 \)

b) \( (x+3)^2 + \frac{(y+2)^2}{4} = 1 \)

12. horizontal compression about the y-axis and vertical compression about the x-axis by a factor of \( \frac{1}{2} \), followed by a translation 2 units left and 3 units up. \( x^2 + y^2 + 4x - 6y + 12 = 0 \)

13. B 
14. C 
15. D 
16. 1.5 
17. 0.45
Conic Sections Lesson #6: Transformations of Hyperbolas

Transformations of the Hyperbola $x^2 - y^2 = 1$

The graph of the hyperbola with equation $x^2 - y^2 = 1$ is shown. The curve has asymptotes with slopes of ±1. Sketch the asymptotes on the grid. Complete the following for the graph.

- domain ___________________
- range ___________________
- coordinates of centre ______
- coordinates of vertices __________
- the length of the transverse axis
  (i.e. distance between vertices) _____

Part 1A Transforming the Hyperbola using Stretches

$x$ is replaced by $\frac{1}{2}x$ and $y$ is replaced by $\frac{1}{3}y$ to get the equation

$$\left(\frac{1}{2}x\right)^2 - \left(\frac{1}{3}y\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{4} - \frac{y^2}{9} = 1.$$

a) Complete the following for the graph of the equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

- The transformation from $x^2 - y^2 = 1$ is

b) The transformed image is shown on the grid. Complete the following.

- domain ___________________
- range ___________________
- coordinates of centre ______
- coordinates of vertices __________
- distance between vertices _____

Note that the slopes of the asymptotes are now $\pm \frac{3}{2}$. 

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Part 1B  Transforming the Hyperbola with Stretches and Translations

\[ \frac{(x - 5)^2}{4} - \frac{(y - 4)^2}{9} = 1. \]

**a)** Complete the following for the graph of the equation \( \frac{(x - 5)^2}{4} - \frac{(y - 4)^2}{9} = 1. \)

- The transformation from \( \frac{x^2}{4} - \frac{y^2}{9} = 1 \) is

**b)** Label the transformed image which is shown on the grid and complete.

- domain _________________
- range _________________
- centre ______
- vertices ____________________
- distance between vertices (length of transverse axis) _____

**Features of the Graph of the Hyperbola \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \)**

The hyperbola defined by the equation \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \) opens along the horizontal axis and has:

- centre \((h, k)\)
- domain: \( x \leq -a + h \) and \( x \geq a + h \)
- range: \( y \in \mathbb{R} \)
- vertices \((-a + h, k)\) and \((a + h, k)\)
- the transverse axis is the line joining the vertices and its length is equal to \(2a\)
- the conjugate axis passes through the centre perpendicular to the transverse axis and its length is equal to \(2b\)
- asymptotes with slopes \(\pm \frac{b}{a}\)
- \(a\) is the horizontal stretch factor and \(b\) is the vertical stretch factor from \(x^2 - y^2 = 1\)
- \(h\) is the horizontal translation and \(k\) is the vertical translation of the centre.
Consider the hyperbola with equation \( \frac{(x + 1)^2}{16} - \frac{9(y - 3)^2}{25} = 1 \).

a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola \( x^2 - y^2 = 1 \).

b) Determine the following features of the graph of \( \frac{(x + 1)^2}{16} - \frac{9(y - 3)^2}{25} = 1 \).
   i) centre  
   ii) distance between vertices  
   iii) coordinates of vertices  
   iv) domain  
   v) range  
   vi) slopes of the asymptotes

c) Use the information above to sketch the graph of \( \frac{(x + 1)^2}{16} - \frac{9(y - 3)^2}{25} = 1 \).

A hyperbola has vertices at (5, 0) and (-5, 0). One of the asymptotes has a slope 2.

a) Find the equation of the hyperbola in standard form.

Extension

b) The hyperbola in a) is transformed by a horizontal expansion by a factor of 4 about the line \( x = -1 \). Determine the equation of the transformed hyperbola.
**Transformations of the Hyperbola** $x^2 - y^2 = -1$

The graph of the hyperbola with equation $x^2 - y^2 = -1$ is shown. The curve has asymptotes with slopes of $\pm 1$.

Sketch the asymptotes on the grid.

Complete the following for the graph.

- **domain:** ________
- **range:** ________________
- **centre:** ________
- **vertices:** ________________
- **the length of the transverse axis** (i.e. distance between vertices) _____

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**Features of the Graph of the Hyperbola** $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = -1$

The hyperbola defined by the equation $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = -1$ **opens along the vertical axis** and has

- **centre** $(h, k)$
- **domain:** $x \in \mathbb{R}$
- **range:** $y \leq -b + k$ and $y \geq b + k$
- **the transverse axis** is the line joining the vertices and its length is equal to $2b$
- **the conjugate axis** passes through the centre perpendicular to the transverse axis and its length is equal to $2a$
- **asymptote with slopes** $\pm \frac{b}{a}$
- **$a$** is the horizontal stretch factor and $b$ is the vertical stretch factor from $x^2 - y^2 = -1$.
- **$h$** is the horizontal translation and $k$ is the vertical translation of the centre.
Consider the hyperbola with equation \( 9(x + 1)^2 - \frac{(y + 2)^2}{9} = -1. \)

a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola \( x^2 - y^2 = -1. \)

b) Determine the following features of the graph of \( 9(x + 1)^2 - \frac{(y + 2)^2}{9} = -1. \)
   i) centre
   ii) distance between vertices
   iii) coordinates of vertices
   iv) domain
   v) range
   vi) slopes of the asymptotes
   vii) \( x \)- and \( y \)-intercepts (to nearest tenth).

c) Use the information above to sketch the graph of \( 9(x + 1)^2 - \frac{(y + 2)^2}{9} = -1. \)
A rectangular hyperbola is a hyperbola of the form \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = \pm 1 \) in which the length of the transverse axis is equal to the length of the conjugate axis, i.e. the values of \( a \) and \( b \) are equal.

The standard form equation of a rectangular hyperbola is \( \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{a^2} = \pm 1 \).

The slopes of the asymptotes are \( \pm \frac{b}{a} = \pm \frac{a}{a} = \pm 1 \), so the asymptotes are at right angles to each other. The hyperbolas \( x^2 - y^2 = 1 \) and \( x^2 - y^2 = -1 \) used in the explorations are examples of rectangular hyperbolas.

Consider the rectangular hyperbola whose vertices are (5, 3) and (11, 3)

a) Determine the equation of the rectangular hyperbola in general form.

b) Determine the equations of the asymptotes of the rectangular hyperbola.

**Assignment**

1. Determine the equation of the given hyperbola after the transformation.
   
a) Determine the equation of the image of the hyperbola \( \frac{(x - 9)^2}{25} - \frac{y^2}{144} = 1 \) after a translation 12 units up and 5 units left.

b) Determine the equation of the image of the hyperbola \( \frac{x^2}{9} - \frac{(y + 2)^2}{16} = 1 \) after a horizontal expansion about the line \( x = 0 \) by a factor 3.
Extension e) Determine the equation of the image of the hyperbola \((x - 1)^2 - \frac{y^2}{4} = -1\) after a vertical expansion of factor 2 about the line \(y = 3\).

2. Consider the hyperbola with equation \(\frac{(x - 5)^2}{64} - \frac{(y + 3)^2}{81} = 1\).
   a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola \(x^2 - y^2 = 1\).

b) Determine the following features of the graph of \(\frac{(x - 5)^2}{64} - \frac{(y + 3)^2}{81} = 1\).
   i) centre
   ii) length of transverse axis
   iii) coordinates of vertices
   iv) domain
   v) range
   vi) slopes of the asymptotes

c) Use the information above to sketch the graph of \(\frac{(x - 5)^2}{64} - \frac{(y + 3)^2}{81} = 1\).
3. A hyperbola has vertices at $(-7, 0)$ and $(7, 0)$. One of the asymptotes has a slope $\frac{3}{2}$. Find the equation of the hyperbola in general form.

4. A hyperbola has vertices at $(-3, 2)$ and $(9, 2)$. The asymptotes have slopes of $\pm \frac{1}{3}$.
   a) Find the equation of the hyperbola in standard form.

   b) The hyperbola is compressed horizontally by a factor of $\frac{2}{3}$ about the line $x = 9$. Determine the equation of the transformed hyperbola.

   c) Determine the equations of the asymptotes of the transformed hyperbola in b).

5. Consider the hyperbola with equation $16(x + 2)^2 - 4(y - 6)^2 = -1$.
   a) Use transformations to describe how the graph of this conic section can be obtained from the graph of the hyperbola $x^2 - y^2 = -1$. 
b) Determine the following features of the graph of \( 16(x + 2)^2 - 4(y - 6)^2 = -1 \).

i) centre  
ii) length of transverse axis  
iii) coordinates of vertices  
iv) domain  
v) range  
vi) slopes of the asymptotes  

c) Use the information above to sketch the graph of \( 16(x + 2)^2 - 4(y - 6)^2 = -1 \).

6. A translation of \( p \) units right and \( q \) units up can be described by the ordered pair \((p, q)\).

a) Determine the equation of the hyperbola \( \frac{(y - 2)^2}{50} - \frac{x^2}{25} = 1 \) after a translation described by the ordered pair \((1, -5)\).

b) The point \( P(5, 12) \) lies on the original hyperbola. Determine the coordinates of \( P' \), the image of \( P \), under the transformation in a).

c) Determine the equations of the asymptotes of the transformed hyperbola in the form \( y - y_1 = m(x - x_1) \).

Multiple Choice 7. A hyperbola has asymptotes with slopes \( \pm \frac{4}{3} \). If the vertices are \((0, -8)\) and \((0, 8)\) the equation of the hyperbola is

A. \( \frac{x^2}{36} - \frac{y^2}{64} = 1 \)
B. \( \frac{x^2}{36} - \frac{y^2}{64} = -1 \)
C. \( \frac{x^2}{9} - \frac{y^2}{16} = 1 \)
D. \( \frac{x^2}{9} - \frac{y^2}{16} = -1 \)
8. The domain of the quadratic relation \( \frac{(y - 8)^2}{64} - \frac{(x - 2)^2}{4} = 1 \) is
   A. \( x \leq -2 \) and \( x \geq 6 \)
   B. \( x \leq 0 \) and \( x \geq 4 \)
   C. \( x \leq -4 \) and \( x \geq 0 \)
   D. \( x \in \mathbb{R} \)

9. A rectangular hyperbola is centred at the origin. Which of the following equations is a possible equation for one of the asymptotes of the hyperbola?
   A. \( y = x + 1 \)  
   B. \( y = -x \)  
   C. \( \frac{y}{4} - \frac{x}{4} = 1 \)  
   D. \( y = 2x \)

10. The slopes of the asymptotes of the hyperbola \( \frac{4(x + 6)^2}{9} - \frac{(y - 1)^2}{9} = 1 \) are \( \pm k \), where \( k > 0 \). The value of \( k \), to the nearest tenth, is _____ .

Answer Key

1. a) \( \frac{(x - 4)^2}{25} - \frac{(y - 12)^2}{144} = 1 \)  
   b) \( \frac{x^2}{81} - \frac{(y + 2)^2}{16} = 1 \)  
   c) \( (x - 1)^2 - \frac{(y + 3)^2}{16} = -1 \)

2. a) horizontal expansion by a factor of 8 about the \( y \)-axis, a vertical expansion by a factor of 9 about the \( x \)-axis, followed by a translation 5 units right and 3 units down.
   b) i) \((5, -3)\)  
      ii) \(16\)  
      iii) \((-3, -3) \) and \((13, -3)\)  
      iv) \( x \leq -3 \) and \( x \geq 13 \)  
      v) \( y \in \mathbb{R} \)  
      vi) \( \pm \frac{9}{8} \)

3. \( 9x^2 - 4y^2 - 441 = 0 \)

4. a) \( \frac{(x - 3)^2}{36} - \frac{(y - 2)^2}{4} = 1 \)  
   b) \( \frac{(x - 5)^2}{16} - \frac{(y - 2)^2}{4} = 1 \)  
   c) \( x + 2y - 9 = 0, \ x - 2y - 1 = 0 \)

5. a) horizontal compression by a factor of \( \frac{1}{4} \) about the \( y \)-axis, a vertical compression by a factor of \( \frac{1}{2} \) about the \( x \)-axis, followed by a translation 2 units left and 6 units up.
   b) i) \((-2, 6)\)  
      ii) \(1\)  
      iii) \((-2, \frac{11}{2})\) and \((-2, \frac{13}{2})\)  
      iv) \( x \in \mathbb{R} \)  
      v) \( y \leq \frac{11}{2} \) and \( y \geq \frac{13}{2} \)  
      vi) \( \pm 2 \)

6. a) \( \frac{(y + 3)^2}{50} - \frac{(x - 1)^2}{25} = 1 \)  
   b) \((6, 7)\)  
   c) \( y + 3 = \pm \sqrt{2} (x - 1) \)


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Sometimes mathematicians have a habit of studying topics to keep their skills sharp or just for fun. At the time, some of these topics may appear to have little practical us, but then many years or even centuries later, these topics turn out to have great scientific value.

Appollonius’ study of conic sections is such a topic. His work with conic sections a large number of applications in our current society.

- Bodies projected upward and obliquely to the pull of gravity in nature (such as the path of a golf ball after it has been struck by a golf club, the design of a headlight of a car or enlarger bulb) and the design of parabolic mirrors in telescopes may be approximated by a parabola. The largest parabolic mirror used in a telescope (approximately 6 m in diameter) is located in the Caucasus mountains of Russia and was built in 1967. However, a company, UPC, is currently building a telescope with a parabolic mirror of 10 m in length in Europe.

- The path of Halley’s Comet, the light path of lithotripsy (a medical procedure for treating kidney stones), and many building designs all follow the path of an ellipse.

- Planes or ships at sea may use LORAN, a navigation system which uses electronic signals in a hyperbolic path to determine the location of a ship or plane. Circular cones intersected by a plane parallel to the axis such as sharpening a pencil or a sonic boom shock wave from a jet follow part of the path of a hyperbola. A hyperbolic path is also used in building designs such as the Saddledome in Calgary.

A special enlarger bulb is designed to enlarge photographs from a 4 x 5 enlarger so that the reflector takes the shape of a parabola if viewed from its side. The diameter of the reflector is 6 cm and the depth of the reflector is 2 cm.

Find the equation of the parabola in standard form if the vertex of the parabola is located at (0, 0) and it opens to the left.
For security reasons, a military plane is designated to fly curved air routes to transport top secret material across the ocean between islands. During one of these flights a pilot is instructed to fly part of a hyperbolic path between two islands passing over an oil rig equidistant from both islands.

If the domain of the flight path is \( \{x \mid 0 \leq x \leq 40\} \) and the range is \( \{y \mid -150 \leq y \leq 150\} \) determine the equation of the hyperbolic path in standard form, where \( a = 10 \).

### Assignment

1. Use the information from Class Ex. #2 to answer the following.

   a) As an alternate route, a pilot is instructed to fly a parabolic path between the islands passing over the oil rig. Determine the equation of the parabola in standard form.

   b) As a third alternate route a pilot has been instructed to fly a semi-elliptical path with centre \((40, 0)\) between the islands passing over the oil rig. Determine the equation of the semi-ellipse in standard form.
c) The coordinate system is changed so that the centre of the ellipse in b) is (0, 0). How can we use the answer in b) to determine the new equation of the semi-ellipse? Determine the new equation of the semi-ellipse in standard form.

2. A competitive ice skating facility is to be designed for the upcoming championships within a 120 m x 90 m wide rectangular area. The design committee is considering a design with semi-circles at each end of the rectangle.

![Diagram of 120 m x 90 m rectangle with semi-circles at each end]

a) Determine the equation, in general form, of both semi-circles using the origin at the centre of the rectangle. State the domain and range for each semi-circle.

b) The committee is also considering an elliptical ice surface. Determine the equation of the largest ellipse possible.
3. A bridge with a curved arch support is to be constructed over a small river. The curved arch support of the bridge is to be 10 metres high and 16 metres wide.

a) If the origin of the coordinate system is taken at the extreme left edge of the curved support, determine the equation of the curve in standard form if it is to be constructed in the form of;
   i) a semi-ellipse
   ii) a parabola.
   iii) the lower branch of a hyperbola where $b = 4$. 

Conic Sections Lesson #7: Applications of Conic Sections
b) A loaded rectangular barge 7 metres high above the water and 8 metres wide will be travelling under the bridge after it is constructed. Determine which of the three curved arch supports in a) will allow the barge to pass safely under the bridge. Calculate the clearance in each case to the nearest tenth of a metre.
4. When the XL-17 jet breaks the sound barrier, the shock wave that is produced at the surface of the earth is hyperbolic in shape.

Determine the equation of the hyperbola in standard form if the slope of one of the asymptotes of the hyperbola is \(-\frac{3}{2}\), the centre is located at \((0, 0)\) and the vertex is located at \((-1500, 0)\).

5. The Musical Arts Entertainment company is constructing a bandshell to obtain a high quality sound for it musical shows. The bandshell is to be constructed in a hyperbolic shape as shown.

Using the vertex as the origin determine the equation of the hyperbola in standard form. State the domain and range.

**Answer Key**

1. a) \( x = \frac{2}{1125} y^2 \)
   
   b) \( \frac{(x - 40)^2}{1600} + \frac{y^2}{22500} = 1 \)
   
   c) \( \frac{x^2}{1600} + \frac{y^2}{22500} = 1 \)

2. a) Left semi-circle \( x^2 + y^2 + 30x - 1800 = 0 \) Right semi-circle \( x^2 + y^2 - 30x - 1800 = 0 \)

   Domain: \( \{ x \mid -60 \leq x \leq -15, x \in \mathbb{R} \} \)
   
   Range: \( \{ y \mid -45 \leq y \leq 45, y \in \mathbb{R} \} \)

   b) \( \frac{x^2}{3600} + \frac{y^2}{2025} = 1 \)

3. a) i) \( \frac{(x - 8)^2}{64} + \frac{y^2}{100} = 1 \) ii) \( y - 10 = \frac{5}{32} (x - 8)^2 \) iii) \( \frac{45(x - 8)^2}{256} - \frac{(y - 14)^2}{16} = -1 \)

   b) Ellipse: yes by 1.7 m Parabola: yes by 0.5 m Hyperbola: No by 0.8 m

4. \( \frac{x^2}{250 000} - \frac{y^2}{5 062 500} = 1 \)

5. \( \frac{19x^2}{2205} - \frac{(y - 7)^2}{49} = -1 \) Domain: \( \{ x \mid -15 \leq x \leq -15, x \in \mathbb{R} \} \)
   
   Range: \( \{ y \mid -5 \leq y \leq 0, y \in \mathbb{R} \} \)
Conics Lesson #8: Converting From General Form To Standard Form

Warm-Up #1

Quadratic relations, or conics, can be written in two forms:

- general form: \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \), where \( A, C, D, E, F \in \mathbb{R} \)

or

- standard form:
  \[
  \frac{(x - h)^2}{a^2} \pm \frac{(y - k)^2}{b^2} = \pm 1
  \]
  
  \[
  y - k = a(x - h)^2
  \]
  
  \[
  x - h = a(y - k)^2
  \]

In order to convert equations of conics sections from general to standard form we will apply the method of completing the square learned in earlier math courses.

Completing the Square

Recall the instructions shown used for the method of completing the square.

Step 1
Factor the leading coefficient out of the \( x^2 \) term and the \( x \) term.

Step 2
Take half of the numerical coefficient of the \( x \) term and square it.

Step 3
Add the squared number in step 2 to both sides of the equation.

Step 4
Simplify and write in factored form.

Class Ex. #1

Consider the equation \( x^2 + 10x - y + 16 = 0 \).

a) Convert \( x^2 + 10x - y + 16 = 0 \) in standard form.

b) Verify the answer in a) use a graphing calculator to graph both equations.

c) Use the standard form to determine the domain, range, and vertex of the parabola.
Class Ex. #2

Consider the quadratic relation with equation \( x^2 - 3y^2 - 10x - 24y - 59 = 0 \).
Convert the equation to standard form and determine the vertices of the conic section.

Class Ex. #3

The equation \( 9x^2 + 25y^2 - 108x + 150y + 324 = 0 \) represents an ellipse.

a) List the following general characteristics of the graph of the ellipse
   - centre
   - vertices
   - domain and range
   - \( x \)- and \( y \)-intercepts

b) Use these results to sketch the graph.
The equation $2x^2 + 2y^2 + 16x - 5y + 8 = 0$ represents a circle.

a) Convert the equation to the form $(x - h)^2 + (y - k)^2 = r^2$.

b) Determine the centre and radius (to the nearest tenth).

Complete Assignment Questions #1 - #9

Assignment

1. Convert the following equations to standard form and determine the type of conic that each represents.
   
   a) $x^2 - 6x - y - 10 = 0$

   b) $x^2 + 3y^2 + 10x - 30y + 91 = 0$
c) \[ 16x^2 - y^2 - 96x + 8y + 112 = 0 \]

d) \[ -2y^2 - x + 20y - 47 = 0 \]

e) \[ 4x^2 - y^2 - 24x + 52 = 0 \]

2. Find the centre and radius of each circle.

a) \[ x^2 + y^2 - 8x - 6y + 9 = 0 \]  
b) \[ x^2 + y^2 - 4x + 3y = 0 \]
3. Consider the conic section with equation $3x^2 - 6x + y - 9 = 0$.
   a) Convert $3x^2 - 6x + y - 9 = 0$ to standard form.

   b) Determine the domain, range, and vertex of the graph of the conic section.

   c) Find the $x$-intercepts and $y$-intercepts and sketch the graph.

4. Consider the equation $16x^2 + 9y^2 + 192x - 36y + 468 = 0$.
   a) Describe the series of transformations applied to the graph of the unit circle $x^2 + y^2 = 1$ which would result in the graph of the equation $16x^2 + 9y^2 + 192x - 36y + 468 = 0$.

   b) Determine the following general characteristics of the graph and sketch the graph:
      - centre
      - vertices
      - domain and range
      - the length of the horizontal diameter
      - $x$- and $y$-intercepts (to the nearest tenth).
5. The centre of the circle with equation $x^2 + y^2 + 2x - 2y - 25 = 0$ is
   A. $(–2, 2)$  B. $(2, –2)$  C. $(-1, 1)$  D. $(1, –1)$

6. The vertex of the parabola with equation $y^2 - 8x - 6y - 7 = 0$ is
   A. $(–2, 3)$  B. $(3, –2)$  C. $(-4, 3)$  D. $(3, –4)$

7. The centre of the ellipse with equation $4x^2 + y^2 - 8x + 4y - 8 = 0$ is
   A. $(-4, 2)$  B. $(4, –2)$  C. $(2, –1)$  D. $(1, –2)$

8. The equation $x^2 - y^2 - 4x + 8y - 21 = 0$ can be written in the form
   $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$.

   The value of $h + k$, to the nearest tenth, is _____.

9. The circle with equation $x^2 + y^2 - 5x - 7 = 0$ has a radius of $k$ units.
   The value of $k$, to the nearest hundredth, is _____.
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